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THE SPECTRUM OF THE EDGE CORONA OF TWO GRAPHS*

YAOPING HOU† AND WAI-CHEE SHIU‡

Abstract. Given two graphs $G_1$, with vertices $1, 2, ..., n$ and edges $e_1, e_2, ..., e_m$, and $G_2$, the edge corona $G_1 \diamond G_2$ of $G_1$ and $G_2$ is defined as the graph obtained by taking $m$ copies of $G_2$ and for each edge $e_k = ij$ of $G_1$, joining edges between the two end-vertices $i, j$ of $e_k$ and each vertex of the $k$-copy of $G_2$. In this paper, the adjacency spectrum and Laplacian spectrum of $G_1 \diamond G_2$ are given in terms of the spectrum and Laplacian spectrum of $G_1$ and $G_2$, respectively. As an application of these results, the number of spanning trees of the edge corona is also considered.

Key words. Spectrum, Adjacency matrix, Laplacian matrix, Corona of graphs.

AMS subject classifications. 05C05, 05C50.

1. Introduction. Throughout this paper, we consider only simple graphs. Let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, ..., n\}$. The adjacency matrix of $G$ denoted by $A(G)$ is defined as $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $i$ and $j$ are adjacent in $G$, 0 otherwise. The spectrum of $G$ is defined as

$$\sigma(G) = (\lambda_1(G), \lambda_2(G), ..., \lambda_n(G)),$$

where $\lambda_1(G) \leq \lambda_2(G) \leq ... \leq \lambda_n(G)$ are the eigenvalues of $A(G)$. The Laplacian matrix of $G$, denoted by $L(G)$ is defined as $D(G) - A(G)$, where $D(G)$ is the diagonal degree matrix of $G$. The Laplacian spectrum of $G$ is defined as

$$S(G) = (\theta_1(G), \theta_2(G), ..., \theta_n(G)),$$

where $0 = \theta_1(G) \leq \theta_2(G) \leq ... \leq \theta_n(G)$ are the eigenvalues of $L(G)$. We call $\lambda_n(G)$ and $\theta_n(G)$ the spectral radius and Laplacian spectral radius, respectively. There is extensive literature available on works related to spectrum and Laplacian spectrum of a graph. See [2, 5, 6] and the references therein to know more.

The corona of two graphs is defined in [4] and there have been some results on the corona of two graphs [3]. The complete information about the spectrum of the corona of two graphs $G, H$ in terms of the spectrum of $G, H$ are given in [1]. In this

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paper, we consider a variation of the corona of two graphs and discuss its spectrum and the number of spanning trees.

**Definition 1.1.** Let $G_1$ and $G_2$ be two graphs on disjoint sets of $n_1$ and $n_2$ vertices, $m_1$ and $m_2$ edges, respectively. The edge corona $G_1 \diamond G_2$ of $G_1$ and $G_2$ is defined as the graph obtained by taking one copy of $G_1$ and $m_1$ copies of $G_2$, and then joining two end-vertices of the $i$-th edge of $G_1$ to every vertex in the $i$-th copy of $G_2$.

Note that the edge corona $G_1 \diamond G_2$ of $G_1$ and $G_2$ has $n_1 + m_1 n_2$ vertices and $m_1 + 2 m_1 n_2 + m_1 m_2$ edges.

**Example 1.2.** Let $G_1$ be the cycle of order 4 and $G_2$ be the complete graph $K_2$ of order 2. The two edge coronas $G_1 \diamond G_2$ and $G_2 \diamond G_1$ are depicted in Figure 1.

![Figure 1: An example of edge corona graphs](image)

Throughout this paper, $G_1$ is assumed to be a connected graph with at least one edge. In this paper, we give a complete description of the eigenvalues and the corresponding eigenvectors of the adjacency matrix of $G_1 \diamond G_2$ when $G_1$ and $G_2$ are both regular graphs and give a complete description of the eigenvalues and the corresponding eigenvectors of the Laplacian matrix of $G_1 \diamond G_2$ for a regular graph $G_1$ and arbitrary graph $G_2$. As an application of these results, we also consider the number of spanning trees of the edge corona.

2. The spectrum of the graph $G_1 \diamond G_2$. Let the vertex set and edge set of a graph $G$ be $V = \{1, 2, ..., n\}$ and $E = \{e_1, e_2, ..., e_m\}$, respectively. The vertex-edge incidence matrix $R(G) = (r_{ij})$ is an $n \times m$ matrix with entry $r_{ij} = 1$ if the vertex $i$ is incident the edge $e_j$ and 0 otherwise.

**Lemma 2.1.** [2, P. 114] Let $G$ be a connected graph with $n$ vertices and $R$ be the vertex-edge incident matrix. Then $\text{rank}(R) = n - 1$ if $G$ is bipartite and $n$ otherwise.

**Lemma 2.2.** [2] Let $G$ be a connected graph with spectral radius $\rho$. Then $-\rho$ is also an eigenvalue of $A(G)$ if and only if $G$ is bipartite. Moreover, if $G$ is a
connected bipartite graph with vertex partition \( V = V_1 \cup V_2 \) and \( X = (X_1, X_2)^T \) is an eigenvector corresponding eigenvalue \( \lambda \) of \( A(G) \) then \( X = (X_1, -X_2)^T \) is an eigenvector corresponding eigenvalue \(-\lambda\) of \( A(G)\).

Let \( A = (a_{ij}) \), \( B \) be matrices. Then the Kronecker product of \( A \) and \( B \) is defined the partition matrix \((a_{ij}B)\) and is denoted by \( A \otimes B \). The row vector of size \( n \) with all entries equal to one is denoted by \( j_n \) and the identity matrix of order \( n \) is denoted by \( I_n \).

Let \( G_1 \) and \( G_2 \) be graphs with \( n_1, n_2 \) vertices and \( m_1, m_2 \) edges, respectively. Then the adjacency matrix of \( G = G_1 \circ G_2 \) can be written as

\[
A(G) = \begin{pmatrix}
A(G_1) & R(G_1) \otimes j_{n_2} \\
(R(G_1) \otimes j_{n_2})^T & I_{m_1} \otimes A(G_2)
\end{pmatrix},
\]

where \( A(G_1) \) and \( A(G_2) \) are the adjacency matrices of the graphs \( G_1 \) and \( G_2 \), respectively, and \( R(G_1) \) is the vertex-edge incidence matrix of \( G_1 \). A complete characterization of the eigenvalues and eigenvectors of \( G_1 \circ G_2 \) will be given when both \( G_1 \) and \( G_2 \) are regular.

Let \( G_1 \) be an \( r_1 \)-regular graph and \( G_2 \) be an \( r_2 \)-regular graph and

\[
\sigma(G_1) = (\mu_1, \mu_2, \ldots, \mu_{n_1}) \quad \text{and} \quad \sigma(G_2) = (\eta_1, \eta_2, \ldots, \eta_{n_2})
\]

be their adjacency spectrum, respectively. If \( G_1 \) is 1-regular then \( G_1 = K_2 \) as \( G_1 \) is connected. In this case, \( G_1 \circ G_2 \) is the complete product of \( K_2 \) and \( G_2 \). By Theorem 2.8 of [2], or by some direct computations, we can obtain the spectrum of \( G = G_1 \circ G_2 \) as

\[
(\eta_1, \ldots, \eta_{n_2-1}, \mu_1 = r_2 + \mu_1 \pm \sqrt{(r_2 - \mu_1)^2 + 4(r_1 + \mu_1)n_2}, \mu_2 \pm \sqrt{(r_2 - \mu_2)^2 + 4(r_1 + \mu_2)n_2}),
\]

where \( \mu_1 = -1, \mu_2 = 1 \) are the spectrum of \( K_2 \).

**Theorem 2.3.** Let \( G_1 \) be an \( r_1 \)-regular \( (r_1 \geq 2) \) graph and \( G_2 \) be an \( r_2 \)-regular graph and their spectra are as in (2.1). Then the spectrum \( \sigma(G) \) of \( G \) is

\[
\begin{pmatrix}
\eta_1 & \eta_2 & \cdots & \eta_{n_2} \\
m_1 & m_1 & \cdots & m_1 - n_1
\end{pmatrix}
=\begin{pmatrix}
r_2 + \mu_1 \pm \sqrt{(r_2 - \mu_1)^2 + 4(r_1 + \mu_1)n_2} & \cdots & \cdots \\
r_2 + \mu_2 \pm \sqrt{(r_2 - \mu_2)^2 + 4(r_1 + \mu_2)n_2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2}
\end{pmatrix}
\]

where entries in the first row are the eigenvalues with the number of repetitions written below, respectively.
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Proof. Let $Z_1, Z_2, ..., Z_{n_2}$ be the orthogonal eigenvectors of $A(G_2)$ corresponding to the eigenvalue $\eta_1, \eta_2, ..., \eta_{n_2} = r_2$, respectively. Note that $G_2$ is $r_2$-regular and $Z_j \perp j$ for $j = 1, 2, ..., n_2 - 1$. Then for $i = 1, 2, ..., m_1$ and for $j = 1, 2, ..., n_2 - 1$, we have (see Figure 2, picture on the left) that $(n_1 + m_1 n_2)$-dimension vectors $(0, 0, ..., 0, Z_j, 0, ..., 0)^T$, where $(i+1)$-th block is $Z_j$ are eigenvectors of $G$ corresponding to eigenvalue $\eta_j$. Thus we obtain $m_1 (n_2 - 1)$ eigenvalues and corresponding eigenvectors of $G$.

Let $X_1, X_2, ..., X_{n_1}$ be the orthogonal eigenvectors of $A(G_1)$ corresponding to the eigenvalues $\mu_1, \mu_2, ..., \mu_{n_1}$, respectively. For $i = 1, 2, ..., n_1$, let

$$\lambda_i = \frac{r_2 + \mu_{n_i} + \sqrt{(r_2 - \mu_{n_i})^2 + 4(r_1 + \mu_{n_i}) n_2}}{2}$$

and

$$\overline{\lambda}_i = \frac{r_2 + \mu_{n_i} - \sqrt{(r_2 - \mu_{n_i})^2 + 4(r_1 + \mu_{n_i}) n_2}}{2}.$$  

Note that $\frac{r_2 + \mu_{n_i} \pm \sqrt{(r_2 - \mu_{n_i})^2 + 4(r_1 + \mu_{n_i}) n_2}}{2} = r_2$ if and only if $\mu_i = -r_1$. So $\lambda_i$ or $\overline{\lambda}_i$ is $r_2$ if and only if $G_1$ is bipartite (note that at most one of $\lambda_i$ is $r_2$). If $G_1$ is bipartite and the bipartition of its vertex set is $V_1 \cup V_2$, then by Lemma 2.2 and some computations, we obtain that $(j, -j, 0, ..., 0)^T$ (1 on $V_1$, -1 on $V_2$, and 0 on all copies of $G_2$ ) is an eigenvector of $G$ corresponding the eigenvalue $-r_1$.

Observe that if $\lambda_i$ and $\overline{\lambda}_i$ are not equal to $r_2$ then $\lambda_i$ and $\overline{\lambda}_i$ are eigenvalues of $G$ corresponding to the eigenvectors $F_i = (X_1, ..., \frac{X_i(s)+X_i(t)}{\lambda_i-r_2} ..., )^T$ and $\overline{F}_i = (X_1, ..., \frac{X_i(s)+X_i(t)}{\overline{\lambda}_i-r_2} ..., )^T$, respectively (see Figure 2, picture in the middle). In fact, it needs only to be checked that characteristic equations $\sum_{v \sim u} F_i(v) = \lambda_i F_i(u)$ (resp. $\sum_{v \sim u} \overline{F}_i(v) = \overline{\lambda}_i \overline{F}_i(u)$) hold for every vertex $u$ in $G$.

For any vertex $u$ in $k$-copy of $G_2$, let edge $e_k = st$, then $F_i(u) = \frac{X_i(s)+X_i(t)}{\lambda_i-r_2}$. Furthermore,

$$\sum_{v \sim u} F_i(v) = r_2 F_i(u) + X_i(s) + X_i(t) = \lambda_i F_i(u).$$
For any vertex $u$ in $G_1$,
\[
\sum_{v \sim u} F_i(v) = \sum_{v \sim u, v \in V(G_1)} F_i(v) + \sum_{v \sim u, v \notin V(G_1)} F_i(v) = \mu_i X_i(u) + r_1 n_2 X_i(u) + \frac{n_2}{\lambda_i - r_2} \sum_{v \sim u, v \in V(G_1)} F_i(v) = \lambda_i X_i(u) = \lambda_i F_i(u).
\]

Therefore we obtain $2n_1$ eigenvalues and corresponding eigenvectors of $G$ if $G_1$ is not bipartite and $2n_1 - 1$ eigenvalues and corresponding eigenvectors of $G$ if $G_1$ is bipartite.

Let $Y_1, Y_2, ..., Y_b$ be a maximal set of independent solution vectors of linear system $R(G_1)Y = 0$. Then $b = m_1 - n_1$ if $G_1$ is not bipartite and $b = m_1 - n_1 + 1$ if $G_1$ is bipartite. For $i = 1, 2, ..., b$, let $H_i = (0, Y_1(e_1)j, ..., Y_1(e_m)j)^T$ (see Figure 2, picture on the right). We can obtain that $H_i$ is an eigenvector of $G$ corresponding to eigenvalues $r_2 = \eta n_2$. Thus these $Y_i'$s provide $b$ eigenvalues and corresponding eigenvectors of $G$.

Therefore we obtain $n_1 + m_1 n_2$ eigenvalues and corresponding eigenvectors of $G$ and it is easy to see that these eigenvectors of $G$ are linearly independent. Hence the proof is completed.

Next we consider the Laplacian spectrum of $G_1 \circ G_2$.

Let $L(G_1)$ and $L(G_2)$ be the Laplacian matrices of the graphs $G_1$ and $G_2$, respectively, and $R(G_1)$ be the vertex-edge incidence matrix of $G_1$. Then the Laplacian matrix of $G = G_1 \circ G_2$ is
\[
L(G) = \begin{pmatrix}
L(G_1) + r_1 n_2 I_{n_1} & -R(G_1) \otimes j_{n_2} \\
-(R(G_1) \otimes j_{n_2})^T & I_{m_1} \otimes (2I_{n_2} + L(G_2))
\end{pmatrix}.
\]

In the following, we give a complete characterization of the Laplacian eigenvalues and eigenvectors of $G_1 \circ G_2$.

Let $G_1$ be an $r_1$-regular graph and $G_2$ be any graph and
\begin{equation}
S(G_1) = (\theta_1, \theta_2, ..., \theta_{n_1}), \quad S(G_2) = (\tau_1, \tau_2, ..., \tau_{n_2})
\end{equation}
be their Laplacian spectra, respectively. If $G_1$ is 1-regular then $G_1 = K_2$ as $G_1$ is connected. In this case, $G_1 \circ G_2$ is the complete product of $K_2$ and $G_2$ (by [5]), or by some direct computations, we can obtain that the Laplacian spectrum of $G = G_1 \circ G_2$
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is

\[ S(G) = (0, \tau_2 + 2, \ldots, \tau_{n_2} + 2, n_2 + 2, n_2 + 2) \, . \]

**Theorem 2.4.** Let \( G_1 \) be an \( r_1 \)-regular (\( r_1 \geq 2 \)) graph and \( G_2 \) be any graph and their Laplacian spectra are written as in (2.2). Let

\[ \beta_i, \bar{\beta}_i = \frac{r_1 n_2 + \theta_i + 2 \pm \sqrt{(r_1 n_2 + \theta_i + 2)^2 - 4(n_2 + 2)\theta_i}}{2} \]

for every \( \theta_i \). Then the Laplacian spectrum \( S(G) \) of \( G \) is

\[ \left( \begin{array}{cccccccc} \tau_1 + 2, & \tau_2 + 2, & \ldots, & \tau_{n_2} + 2, & \beta_1, & \bar{\beta}_1, & \ldots, & \beta_{n_1}, & \bar{\beta}_{n_1} \end{array} \right) \]

where entries in the first row are the eigenvalues with the number of repetitions written below, respectively.

**Proof.** Let \( Z_1, Z_2, \ldots, Z_{n_2} \) be the eigenvectors of \( L(G_2) \) corresponding to the eigenvalues \( 0 = \tau_1, \tau_2, \ldots, \tau_{n_2} \). Note that \( Z_j \perp j \) for \( j = 2, \ldots, n_2 \). Then for \( i = 1, 2, \ldots, m_1 \) and for \( j = 2, 3, \ldots, n_2 \), we have that \( (n_1 + m_1 n_2) \)-dimension vectors \((0, 0, \ldots, 0, Z_j, 0, \ldots, 0)^T \), where \((i+1)\)-th block is \( Z_j \), are eigenvectors of \( L(G) \) corresponding to eigenvalue \( \tau_j + 2 \) (see Figure 3, picture on the left). Thus we obtain \( m_1(n_2 - 1) \) eigenvalues and corresponding eigenvectors of \( L(G) \).

Let \( X_1, X_2, \ldots, X_{n_1} \) be the orthogonal eigenvectors of \( L(G_1) \) corresponding to the eigenvalues \( \theta_1, \theta_2, \ldots, \theta_{n_1} \), respectively. For \( i = 1, 2, \ldots, n_1 \), note that:

\[ \beta_i, \bar{\beta}_i = \frac{r_1 n_2 + \theta_i + 2 \pm \sqrt{(r_1 n_2 + \theta_i + 2)^2 - 4(n_2 + 2)\theta_i}}{2} = \frac{r_1 n_2 + \theta_i + 2 \pm \sqrt{(r_1 n_2 + \theta_i - 2)^2 + 4n_2(2r_1 - \theta_i)}}{2} \]

since \( r_1 \geq 2, n_2 \geq 1, \beta_i \neq 2 \). Note that \( \theta_i \leq 2r_1 \) and the equality holds if and only if \( G_1 \) is bipartite. Note that \( \bar{\beta}_i = 2 \) implies that \( \theta_i = 2r_1 \). That is, \( \bar{\beta}_i = 2 \) appears only if \( G_1 \) is bipartite and \( i = n_1 \). Moreover, if \( G_1 \) is bipartite and the bipartition of its vertex set is \( V_1 \cup V_2 \), then it is easy to check that \((j, -j, 0, \ldots, 0)^T \) (1 on \( V_1 \),

**Figure 3:** Description of Laplacian eigenvectors
We have
\[-1 \text{ on } V_2, \text{ and } 0 \text{ on all copies of } G_2 \text{ is an eigenvector corresponding the eigenvalue } (n_1 + 2)r_1 = \beta_{n_1} \text{ of } L(G).\]

Observe that if \( \beta_i \) and \( \bar{\beta}_i \) are not equal to 2, then \( \beta_i \) and \( \bar{\beta}_i \) are eigenvalues of \( L(G) \) and 
\[F_i = (X_i, ..., \frac{X_i(s) + X_i(t)}{2 - \beta_i}, ...)^T \text{ and } \tilde{F}_i = (X_i, ..., \frac{X_i(s) + X_i(t)}{2 - \bar{\beta}_i}, ...)^T \]
are eigenvectors of \( \beta_i \) and \( \bar{\beta}_i \) respectively (see Figure 3, picture in the middle). In fact, it needs only to be checked that characteristic equations \( d_G(u)F_i(u) - \sum_{v \sim u} F_i(v) = \beta_i F_i(u) \) (resp. \( d_G(u)\tilde{F}_i(u) - \sum_{v \sim u} \tilde{F}_i(v) = \bar{\beta}_i \tilde{F}_i(u) \)) hold for every vertex \( u \) in \( G \), where \( d_G(u) \) is the degree of the vertex \( u \) in \( G \).

For every vertex \( u \) in \( k \)-copy of \( G_2 \), let the edge \( e_k = st \), then 
\[d_G(u)F_i(u) - \sum_{v \sim u} F_i(v) = d_{G_2}(u) + 2)F_i(u) - d_{G_2}(u)\frac{X_i(s) + X_i(t)}{2 - \beta_i} - (X_i(s) + X_i(t)) \]
\[= \beta_i F_i(u).\]

For every vertex \( u \) in \( G_1 \), note that 
\[r_1X_i(u) - \sum_{v \sim u} X_i(v) = \theta_1X_i(u).\]

We have 
\[d_G(u)F_i(u) - \sum_{v \sim u} F_i(v) = (r_1 + r_1n_2)F_i(u) - \sum_{v \sim u, v \in V(G_1)} F_i(v) + \sum_{v \sim u, v \notin V(G_1)} F_i(v)\]
\[= (r_1 + r_1n_2)X_i(u) - \sum_{v \sim u, v \in V(G_1)} X_i(v) - \sum_{v \sim u, v \notin V(G_1)} \frac{n_2}{2 - \beta_i} (X_i(u) + X_i(v))\]
\[= (r_1 + r_1n_2)(2 - \beta_i) - 2n_2r_1 + n_2\theta_1 \frac{X_i(u)}{2 - \beta_i} + (\theta_1 - r_1)X_i(u)\]
\[= \beta_i X_i(u) = \beta_i F_i(u).\]

Therefore we obtain \( 2n_1 \) eigenvalues and corresponding eigenvectors of \( L(G) \) if \( G_1 \) is not bipartite, and \( 2n_1 - 1 \) eigenvalues and corresponding eigenvectors of \( L(G) \) if \( G_1 \) is bipartite.

Let \( Y_1, Y_2, ..., Y_b \) be a maximal set of independent solution vectors of the linear system \( R(G_1)Y = 0 \). Then \( b = m_1 - n_1 \) if \( G_1 \) is not bipartite, and \( b = m_1 - n_1 + 1 \) if \( G_1 \) is bipartite. For \( i = 1, 2, ..., b \), let \( H_i = (0, Y_i(e_1), ..., Y_i(e_m))^T \) (see Figure 3, picture on the right). We can obtain that \( H_i \) is an eigenvector corresponding the eigenvalue \( 2 = \tau_1 + 2 \) of \( L(G) \). Thus these \( Y_i \)'s provide \( b \) eigenvalues and corresponding eigenvectors of \( L(G) \).
Therefore we obtain \( n_1 + m_1n_2 \) eigenvalues and corresponding eigenvectors of \( L(G) \) and it is easy to see that these eigenvectors of \( L(G) \) are linearly independent. Hence the proof is completed. 

As an application of the above results, we give the number of spanning trees of the edge corona of two graphs.

Let \( G \) be a connected graph with \( n \) vertices and Laplacian eigenvectors \( 0 = \theta_1 < \theta_2 \leq \cdots \leq \theta_n \). Then the number of spanning trees of \( G \) is

\[
t(G) = \frac{\theta_2\theta_3 \cdots \theta_n}{n}.
\]

By Theorem 2.4 we have

**Proposition 2.5.** For a connected \( r_1 \)-regular graph \( G_1 \) and arbitrary graph \( G_2 \), let the number of spanning trees of \( G_1 \) be \( t(G_1) \) and the Laplacian spectra of \( G_2 \) be \( 0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{n_2} \). Then the number of spanning trees of \( G_1 \odot G_2 \) is

\[
t(G_1 \odot G_2) = 2^{m_1-n_1+1}(n_2+2)^{n_1-1}t(G_1)(\tau_2 + 2)^{m_1} \cdots (\tau_{n_2} + 2)^{m_1}.
\]

**Proof.** Following the notions in Theorem 2.4, note that \( \beta_i\beta_i = (n_2 + 2)\theta_i \) for \( i = 1, 2, \ldots, n_1 \) and \( \beta_1 = r_1n_2 + 2, \beta_1 = 0 \). Thus

\[
t(G_1 \odot G_2) = \frac{2^{m_1-n_1}(r_1n_2 + 2)(n_2 + 2)^{n_1-1} \prod_{i=2}^{n_2}(\tau_i + 2)^{m_1} \prod_{j=2}^{r_1} \theta_j}{n_1 + m_1n_2}
= \frac{n_12^{m_1-n_1}(r_1n_2 + 2)(n_2 + 2)^{n_1-1}t(G_1) \prod_{i=2}^{n_2}(\tau_i + 2)^{m_1}}{n_1 + m_1n_2}
= 2^{m_1-n_1+1}(n_2 + 2)^{n_1-1}t(G_1)(\tau_2 + 2)^{m_1} \cdots (\tau_{n_2} + 2)^{m_1}.
\]

The last equality follows from \( n_1 + m_1n_2 = \frac{n_12^{r_1n_2}}{2} \).

By Proposition 2.5, we have \( t(G \odot K_1) = 2^{m-n+1}3^{n-1}t(G) \) for a regular graph \( G \). In fact \( t(G \odot K_1) = 2^{m-n+1}3^{n-1}t(G) \) holds for arbitrary graph \( G \) by the following proposition.

**Proposition 2.6.** Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Then the number of spanning trees of \( G \odot K_1 \) is \( 2^{m-n+1}3^{n-1}t(G) \), where \( t(G) \) is the number of spanning trees of \( G \).

**Proof.** Note that the Laplacian matrix of \( G \odot K_1 \) is

\[
L(G \odot K_1) = \begin{pmatrix}
L(G) + D(G) & -R \\
-R^T & 2I_m
\end{pmatrix}.
\]
Let \((L(G))_{11}\) be the reduced Laplacian matrix of \(G\) obtained by removing the first row and first column of \(L(G)\) and \(R_1\) be the matrix obtained by removing the first row of the vertex-edge incidence matrix \(R\). By the Matrix-Tree theorem [2], we have

\[
t(G \odot K_1) = \det((L(G) + D(G))_{11} - R_1 - R_1^T 2I_m) = 2^m \det\left(\frac{3}{2}(D(G) - A(G))_{11}\right) = 2^{m-n+1}3^{n-1}t(G).
\]

\(t(G \odot K_1) = 2^m \det\left(\frac{3}{2}(D(G) - A(G))_{11}\right) = 2^{m-n+1}3^{n-1}t(G). \]

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REFERENCES