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# THE EIGENVALUE DISTRIBUTION OF BLOCK DIAGONALLY DOMINANT MATRICES AND BLOCK $H-$MATRICES*

CHENG-YI ZHANG†, SHUANGHUA LUO‡, AIQUN HUANG§, AND JUNXIANG LU¶

**Abstract.** The paper studies the eigenvalue distribution of some special matrices, including block diagonally dominant matrices and block $H-$matrices. A well-known theorem of Taussky on the eigenvalue distribution is extended to such matrices. Conditions on a block matrix are also given so that it has certain numbers of eigenvalues with positive and negative real parts.

**Key words.** Eigenvalues, Block diagonally dominant, Block $H-$matrix, Non-Hermitian positive (negative) definite.

**AMS subject classifications.** 15A15, 15F10.

**1. Introduction.** The eigenvalue distribution of a matrix has important consequences and applications (see e.g., [4], [6], [9], [12]). For example, consider the ordinary differential equation (cf. Section 2.0.1 of [4])

$$\frac{dx}{dt} = A[x(t) - \hat{x}], \tag{1.1}$$

where $A \in C^{n\times n}$ and $x(t), \hat{x} \in C^n$. The vector $\hat{x}$ is an equilibrium of this system. It is not difficult to see that $\hat{x}$ of system is globally stable if and only if each eigenvalue of $-A$ has positive real part, which concerns the eigenvalue distribution of the matrix $A$. The analysis of stability of such a system appears in mathematical biology, neural networks, as well as many problems in control theory. Therefore, there is considerable interest in the eigenvalue distribution of some special matrices $A$ and some results are classical. For example, Taussky in 1949 [14] stated the following theorem.

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THEOREM 1.1. ([14]) *Let* $A = (a_{ij}) \in C^{n\times n}$ *be strictly or irreducibly diagonally dominant with positive real diagonal entries* $a_{ii}$ *for all* $i \in N = \{1, 2, \cdots, n\}$*. Then for arbitrary eigenvalue* $\lambda$ *of* $A$*, we have* $Re(\lambda) > 0$.

Tong [15] improved Taussky's result in [14] and proposed the following theorem on the eigenvalue distribution of strictly or irreducibly diagonally dominant matrices.

THEOREM 1.2. ([15]) *Let* $A = (a_{ij}) \in C^{n\times n}$ *be strictly or irreducibly diagonally dominant with real diagonal entries* $a_{ii}$ *for all* $i \in N$*. Then* $A$ *has* $|J_+(A)|$ *eigenvalues with positive real part and* $|J_-(A)|$ *eigenvalues with negative real part, where* $J_+(A) = \{i \mid a_{ii} > 0,\ i \in N\},\ J_-(A) = \{i \mid a_{ii} < 0,\ i \in N\}$.

Later, Jiaju Zhang [23], Zhaoyong You et al. [18] and Jianzhou Liu et al. [6] extended Tong's results in [15] to conjugate diagonally dominant matrices, generalized conjugate diagonally dominant matrices and $H-$matrices, respectively. Liu's result is as follows.

THEOREM 1.3. ([6]) *Let* $A = (a_{ij}) \in H_n$ *with real diagonal entries* $a_{ii}$ *for all* $i \in N$*. Then* $A$ *has* $|J_+(A)|$ *eigenvalues with positive real part and* $|J_-(A)|$ *eigenvalues with negative real part.*

Recently, Cheng-yi Zhang et al. ([21], [22]) generalized Tong's results in [15] to nonsingular diagonally dominant matrices with complex diagonal entries and established the following conclusion.

THEOREM 1.4. ([21], [22]) Given a matrix $A \in C^{n\times n}$, if $\widehat{A}$ is nonsingular diagonally dominant, where $\widehat{A} = (\widehat{a}_{ij}) \in C^{n\times n}$ is defined by

$$\widehat{a}_{ij} = \begin{cases} Re(a_{ii}), & \text{if } i = j, \\ a_{ij}, & \text{if } i \neq j, \end{cases}$$

then $A$ has $|J_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)|$ eigenvalues with negative real part, where $J_{R_+}(A) = \{i \mid Re(a_{ii}) > 0,\ i \in N\},\ J_{R_-}(A) = \{i \mid Re(a_{ii}) < 0,\ i \in N\}$.

However, there exists a dilemma in practical application. That is, for a large-scale matrix or a matrix which is neither a diagonally dominant matrix nor an $H-$matrix, it is very difficult to obtain the property of this class of matrices.

On the other hand, David G. Feingold and Richard S. Varga [2], Zhao-yong You and Zong-qian Jiang [18] and Shu-huang Xiang [17], respectively, generalized the concept of diagonally dominant matrices and proposed two classes of block diagonally dominant matrices, i.e., $I-$block diagonally dominant matrices [2] and II$-$block diagonally dominant matrices [14], [15]. Later, Ben Polman [10], F. Robert [11], Yong-zhong Song [13], L.Yu. Kolotilina [5] and Cheng-yi Zhang and Yao-tang Li [20]

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also presented two classes of block $H-$matrices such as $I-$block $H-$matrices[11] and II$-$block $H-$matrices[10] on the basis of the previous work.

It is known that a block diagonally dominant matrix is not always a diagonally dominant matrix (an example is seen in [2, (2.6)]). So suppose a matrix $A$ is not strictly (or irreducibly) diagonally dominant nor an $H-$matrix. Using and appropriate partitioning of $A$, can we obtain its eigenvalue distribution when it is block diagonally dominant or a block $H-$matrix?

David G. Feingold and Richard S. Varga (1962) showed that an $I-$block strictly or irreducibly diagonally dominant diagonally dominant matrix has the same property as the one in Theorem 1.1. The result reads as follows.

Theorem 1.5. ([2]) *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ be $I-$block strictly or irreducibly diagonally dominant with all the diagonal blocks being $M-$matrices. Then for arbitrary eigenvalue $\lambda$ of $A$, we have $Re(\lambda) > 0$.*

The purpose of this paper is to establish some theorems on the eigenvalue distribution of block diagonally dominant matrices and block H-matrices. Following the result of David G. Feingold and Richard S. Varga, the well-known theorem of Taussky on the eigenvalue distribution is extended to block diagonally dominant matrices and block $H-$matrices with each diagonal block being non-Hermitian positive definite. Then, the eigenvalue distribution of some special matrices, including block diagonally dominant matrices and block $H$-matrices, is studied further to give conditions on the block matrix $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ such that the matrix $A$ has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part; here $J_P^+(A)$ $(J_P^-(A))$ denotes the set of all indices of non-Hermitian positive (negative) definite diagonal blocks of $A$ and $n_k$ is the order of the diagonal block $A_{kk}$ for $k \in J_P^+(A) \cup J_P^-(A)$.

The paper is organized as follows. Some notation and preliminary results about special matrices including block diagonally dominant matrices and block $H-$matrices are given in Section 2. The theorem of Taussky on the eigenvalue distribution is extended to block diagonally dominant matrices and block $H$-matrices in Section 3. Some results on the eigenvalue distribution of block diagonally dominant matrices and block $H$-matrices are then presented in Section 4. Conclusions are given in Section 5.

**2. Preliminaries.** In this section we present some notions and preliminary results about special matrices that are used in this paper. Throughout the paper, we denote the conjugate transpose of the vector $x$, the conjugate transpose of the matrix $A$, the spectral norm of the matrix $A$ and the cardinality of the set $\alpha$ by $x^H$, $A^H$, $\|A\|$ and $|\alpha|$, respectively. $C^{m\times n}$ $(R^{m\times n})$ will be used to denote the set of all $m \times n$ complex (real) matrices. Let $A = (a_{ij}) \in R^{m\times n}$ and $B = (b_{ij}) \in R^{m\times n}$, we write $A \geq B$,

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if $a_{ij} \geq b_{ij}$ holds for all $i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$. A matrix $A = (a_{ij}) \in R^{n\times n}$ is called a $Z-$matrix if $a_{ij} \leq 0$ for all $i \neq j$. We will use $Z_n$ to denote the set of all $n \times n$ $Z-$matrices. A matrix $A = (a_{ij}) \in R^{n\times n}$ is called an $M-$matrix if $A \in Z_n$ and $A^{-1} \geq 0$. $M_n$ will be used to denote the set of all $n \times n$ $M-$matrices.

The comparison matrix of a given matrix $A = (a_{ij}) \in C^{n\times n}$, denoted by $\mu(A) = (\mu_{ij})$, is defined by

$$\mu_{ij} = \begin{cases} |a_{ii}|, & \text{if } \ i = j, \\ -|a_{ij}|, & \text{if } \ i \neq j. \end{cases}$$

It is clear that $\mu(A) \in Z_n$ for a matrix $A \in C^{n\times n}$. A matrix $A \in C^{n\times n}$ is called $H-$matrix if $\mu(A) \in M_n$. $H_n$ will denote the set of all $n \times n$ $H-$matrices.

A matrix $A \in C^{n\times n}$ is called Hermitian if $A^H = A$ and skew-Hermitian if $A^H = -A$. A Hermitian matrix $A \in C^{n\times n}$ is called Hermitian positive definite if $x^H Ax > 0$ for all $0 \neq x \in C^n$ and Hermitian negative definite if $x^H Ax < 0$ for all $0 \neq x \in C^n$. A matrix $A \in C^{n\times n}$ is called non-Hermitian positive definite if $Re(x^H Ax) > 0$ for all $0 \neq x \in C^n$ and non-Hermitian negative definite if $Re(x^H Ax) < 0$ for all $0 \neq x \in C^n$. Let $A \in C^{n\times n}$, then $H = (A + A^H)/2$ and $S = (A - A^H)/2$ are called the Hermitian part and the skew-Hermitian

part of the matrix $A$, respectively. Furthermore, $A$ is non-Hermitian positive (negative) definite if and only if $H$ is Hermitian positive (negative) definite (see [3,7,8]).

Let $x = (x_1, x_2, \cdots, x_n)^T \in C^n$. The Euclidean norm of the vector $x$ is defined by $\|x\| = \sqrt{(x^H x)} = \sqrt{\sum_{i=1}^n |x_i|^2}$ and the spectral norm of the matrix $A \in C^{n\times n}$ is defined by

$$\|A\| = \sup_{0\neq x \in C^n} \left( \frac{\|Ax\|}{\|x\|} \right) = \sup_{\|y\|=1} \left( \|Ay\| \right). \tag{2.1}$$

If $A$ is nonsingular, it is useful to point out that

$$\|A^{-1}\|^{-1} = \inf_{0\neq x \in C^n} \left( \frac{\|Ax\|}{\|x\|} \right). \tag{2.2}$$

With (2.2), we can then define $\|A^{-1}\|^{-1}$ by continuity to be zero whenever $A$ is singular. Therefore, for $B \in C^{n\times n}$ and $0 \neq C \in C^{n\times m}$,

$$\|B^{-1}C\|^{-1} = \inf_{0\neq x \in C^n} \left( \frac{\|B^H x\|}{\|C^H x\|} \right) \tag{2.3}$$

if $B$ is nonsingular, and

$$\|B^{-1}C\|^{-1} \to 0 \Rightarrow \|B^{-1}C\| \to \infty \tag{2.4}$$

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by continuity if $B$ is singular.

A matrix $A \in C^{n\times n}(n \geq 2)$ is called reducible if there exists an $n\times n$ permutation matrix $P$ such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where $A_{11} \in C^{r\times r}$, $A_{22} \in C^{(n-r)\times(n-r)}$, $1 \leq r < n$. If no such permutation matrix exists, then $A$ is called irreducible. $A = (a_{11}) \in C^{1\times 1}$ is irreducible if $a_{11} \neq 0$, and reducible, otherwise.

A matrix $A = (a_{ij}) \in C^{n\times n}$ is diagonally dominant by row if

$$|a_{ii}| \geq \sum_{j=1,j\neq i}^{n} |a_{ij}| \tag{2.5}$$

holds for all $i \in N = \{1, 2, \cdots, n\}$. If inequality in (2.5) holds strictly for all $i \in N$, $A$ is called strictly diagonally dominant by row; if $A$ is irreducible and diagonally dominant with inequality (2.5) holding strictly for at least one $i \in N$, $A$ is called irreducibly diagonally dominant by row.

By $D_n$, $SD_n$ and $ID_n$ denote the sets of matrices which are $n\times n$ diagonally dominant, $n \times n$ strictly diagonally dominant and $n \times n$ irreducibly diagonally dominant, respectively.

Let $A = (a_{ij}) \in C^{n\times n}$ be partitioned as the following form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{bmatrix}, \tag{2.6}$$

where $A_{ll}$ is an $n_l\times n_l$ nonsingular principal submatrix of $A$ for all $l \in S = \{1, 2, \cdots, s\}$ and $\sum_{l=1}^{s} n_l = n$. By $C_s^{n\times n}$ denote the set of all $s \times s$ block matrices in $C^{n\times n}$ partitioned as (2.6). Note: $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ implies that each diagonal block $A_{ll}$ of the block matrix $A$ is nonsingular for all $l \in S$.

Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ and $S = \{1, 2, \cdots, s\}$, we define index sets

$$J_P^+(A) = \{i \mid A_{ii}\ is\ non-Hermitian\ positive\ definite,\ i \in S\},$$
$$J_P^-(A) = \{i \mid A_{ii}\ is\ non-Hermitian\ negative\ definite,\ i \in S\}.$$

From the index sets above, we know that $J_P^+(A) = S$ shows that each diagonal block $A_{ll}$ of $A$ is non-Hermitian positive definite for all $l \in S$ and $J_P^+(A) \cup J_P^-(A) = S$

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shows that each diagonal block $A_{ll}$ of $A$ is either non-Hermitian positive definite or non-Hermitian negative definite for all $l \in S$.

Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$. Then $A$ is called block irreducible if either its $I-$block comparison matrix $\mu_I(A) = (w_{lm}) \in R^{s\times s}$ or its $\text{II}-$block comparison matrix $\mu_{\text{II}}(A) = (m_{lm}) \in R^{s\times s}$ is irreducible, where

$$(2.7)\quad w_{lm} = \begin{cases} \|A_{ll}^{-1}\|^{-1}, & l = m \\ -\|A_{lm}\|, & l \neq m \end{cases}, \quad m_{lm} = \begin{cases} 1, & l = m \\ -\parallel A_{ll}^{-1}A_{lm} \parallel, & l \neq m \end{cases}.$$

A block matrix $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ is called $I-$block diagonally dominant if its $I-$block comparison matrix comparison matrix, $\mu_I(A) \in D_s$. If $\mu_I(A) \in SD_s$, $A$ is $I-$block strictly diagonally dominant; and if $\mu_I(A) \in ID_s$, $A$ is called $I-$block irreducibly diagonally dominant.

Similarly, a block matrix $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ is called $\text{II}-$block diagonally dominant if its $\text{II}-$block comparison matrix, $\mu_{\text{II}}(A) \in D_s$. If $\mu_{\text{II}}(A) \in SD_s$, $A$ is $\text{II}-$block strictly diagonally dominant; and if $\mu_{\text{II}}(A) \in ID_s$, $A$ is called $\text{II}-$block irreducibly diagonally dominant.

A block matrix $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ is called an $I-$block $H-$matrix (resp., a $\text{II}-$block $H-$matrix) if its $I-$block comparison matrix $\mu_I(A) = (w_{lm}) \in R^{s\times s}$ (resp., its $\text{II}-$block comparison matrix $\mu_{\text{II}}(A) = (m_{lm}) \in R^{s\times s}$) is an $s \times s$ $M-$matrix.

In the rest of this paper, we denote the set of all $s\times s$ $I-$block (strictly, irreducibly) diagonally dominant matrices, all $s \times s$ $\text{II}-$block (strictly, irreducibly) diagonally dominant matrices, all $s\times s$ $I-$block $H-$matrices and all $s\times s$ $\text{II}-$block $H-$matrices by $IBD_s(IBSD_s,\ IBID_s)$, $\text{II}BD_s(\text{II}BSD_s,\ \text{II}BID_s)$, $IBH_s$ and $\text{II}BH_s$, respectively.

It follows that we will give some lemmas to be used in the following sections.

Lemma 2.1. (see [2,5]) *If A block matrix $A = (A_{lm})_{s\times s} \in IBSD_s \cup IBID_s$, then $A$ is nonsingular.*

Lemma 2.2. $IBSD_s \cup IBID_s \subset IBH_s$ and $\text{II}BSD_s \cup \text{II}BID_s \subset \text{II}BH_s$

*Proof.* According to the definition of $I-$block strictly or irreducibly diagonally dominant matrices, $\mu_I(A) \in SD_s \cup ID_s$ for any block matrix $A \in IBSD_s \cup IBID_s$. Since $SD_s \cup ID_s \subset H_s$ (see Lemma 2.3 in [21]), $\mu_I(A) \in H_s$. As a result, $A \in IBH_s$ coming from the definition of $I-$block $H-$matrices. Therefore, $IBSD_s \cup IBID_s \subset IBH_s$. Similarly, we can prove $\text{II}BSD_s \cup \text{II}BID_s \subset \text{II}BH_s$. ◻

Lemma 2.3. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$. Then $A \in \text{II}BH_s$ $(IBH_s)$ if and only if there exists a block diagonal matrix $D = diag(d_1 I_1, \cdots, d_s I_s)$, where $d_l > 0$, $I_l$ is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum_{l=1}^s n_l = n$, such that $AD \in \text{II}BSD_s$ $(IBSD_s)$.*

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*Proof.* Using Theorem 6.2.3 ($M_{35}$) in [1, pp.136-137], the conclusion of this lemma is obtained immediately. □

LEMMA 2.4. (see [7]) *If a matrix $A \in C^{n\times n}$ is non-Hermitian positive definite, then for arbitrary eigenvalue $\lambda$ of $A$, we have $Re(\lambda) > 0$.*

LEMMA 2.5. (see [16]) *Let $A \in C^{n\times n}$. Then*

$$\|A\| = \rho(A^H A), \tag{2.8}$$

*the spectral radius of $A^H A$. In particular, if $A$ is Hermitian, then $\|A\| = \rho(A)$.*

**3. Some generalizations of Taussky's theorem.** In this section, the famous Taussky's theorem on the eigenvalue distribution is extended to block diagonally dominant matrices and block $H$-matrices. The following lemmas will be used in this section.

LEMMA 3.1. (see [8]) *Let $A = (a_{ij}) \in C^{n\times n}$ be non-Hermitian positive definite with Hermitian part $H = (A + A^H)/2$. Then*

$$\|A^{-1}\| \leq \|H^{-1}\|. \tag{3.1}$$

LEMMA 3.2. *Let $A \in C^{n\times n}$ be non-Hermitian positive definite with Hermitian part $H = (A+A^H)/2$. Then for arbitrary complex number $\alpha \neq 0$ with $Re(\alpha) \geq 0$, we have*

$$\|(\alpha I + A)^{-1}\| \leq \|H^{-1}\|, \tag{3.2}$$

*where $I$ is the identity matrix and $\|A\|$ is the spectral norm of the matrix $A$.*

*Proof.* Since $A$ is non-Hermitian positive definite, for arbitrary complex number $\alpha \neq 0$ with $Re(\alpha) \geq 0$, we have $\alpha I + A$ is non-Hermitian positive definite. It then follows from Lemma 3.1 that

$$\|(\alpha I + A)^{-1}\| \leq \|[Re(\alpha)I + H]^{-1}\|. \tag{3.3}$$

Since $H$ is Hermitian positive definite, so is $Re(\alpha)I+H$. Thus, the smallest eigenvalue of $Re(\alpha)I + H$ is $\tau(Re(\alpha)I + H) = Re(\alpha) + \tau(H)$. Following Lemma 2.5, we have

$$\begin{aligned} \|[Re(\alpha)I + H]^{-1}\| &= \rho([Re(\alpha)I + H]^{-1}) = \frac{1}{\tau(Re(\alpha)I + H)} \\ &= \frac{1}{Re(\alpha) + \tau(H)} \leq \frac{1}{\tau(H)} = \rho(H^{-1}) \\ &= \|H^{-1}\|. \end{aligned} \tag{3.4}$$

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Then it follows from (3.3) and (3.4) that $\|(\alpha I + A)^{-1}\| \leq \|H^{-1}\|$, which completes the proof. □

Lemma 3.3. (see [2], The generalization of the Gersgorin Circle Theorem) *Let $A \in C^{n\times n}$ be partitioned as* (2.6)*. Then each eigenvalue $\lambda$ of $A$ satisfies that*

$$\|(\lambda I - A_{ll})^{-1}\|^{-1} \leq \sum_{m=1, m\neq l}^{s} \|A_{lm}\| \tag{3.5}$$

*holds for at least one $l \in S$.*

Theorem 3.4. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) = S$. If $\widehat{A} \in IBSD_s$, where $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ is defined by*

$$\widehat{A}_{lm} = \begin{cases} H_{ll} = (A_{ll} + A_{ll}^H)/2, & l = m \\ A_{lm}, & otherwise, \end{cases} \tag{3.6}$$

*then for any eigenvalue $\lambda$ of $A$, we have $Re(\lambda) > 0$.*

*Proof.* The conclusion can be proved by contradiction. Assume that $\lambda$ be any eigenvalue of $A$ with $Re(\lambda) \leq 0$. Following Lemma 3.3, (3.5) holds for at least one $l \in S$. Since $J_P^+(A) = S$ shows that each diagonal block $A_{ll}$ of $A$ is non-Hermitian positive definite for all $l \in S$, it follows from Lemma 3.2 that

$$\|(\lambda I - A_{ll})^{-1}\| \leq \|H_{ll}^{-1}\|$$

and hence

$$\|(\lambda I - A_{ll})^{-1}\|^{-1} \geq \|H_{ll}^{-1}\|^{-1} \tag{3.7}$$

for all $l \in S$. According to (3.5) and (3.7) we have that

$$\|H_{ll}^{-1}\|^{-1} \leq \sum_{m=1, m\neq l}^{s} \|A_{lm}\|$$

holds for at least one $l \in S$. This shows $\mu_I(\widehat{A}) = (w_{lm}) \notin SD_s$. As a result, $\widehat{A} \notin IBSD_s$, which is in contradiction with the assumption $\widehat{A} \in IBSD_s$. Therefore, the conclusion of this theorem holds. □

Theorem 3.5. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) = S$. If $\widehat{A} \in IBH_s$, where $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ is defined in* (3.6)*, then for any eigenvalue $\lambda$ of $A$, we have $Re(\lambda) > 0$.*

*Proof.* Since $\widehat{A} \in IBH_s$, it follows from Lemma 2.3 that there exists a block diagonal matrix $D = diag(d_1 I_1, \cdots, d_s I_s)$, where $d_l > 0$, $I_l$ is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum_{l=1}^{s} n_l = n$, such that $\widehat{A}D \in IBSD_s$, i.e.,

$$\|H_{ll}^{-1}\|^{-1} d_l > \sum_{m=1, m\neq l}^{s} \|A_{lm}\| d_m \tag{3.8}$$

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holds for all $l \in S$. Multiply the inequality (3.8) by $d_l^{-1}$, then

$$\|H_{ll}^{-1}\|^{-1} > \sum_{m=1,m\neq l}^{s} d_l^{-1}\|A_{lm}\|d_m = \sum_{m=1,m\neq l}^{s} \|d_l^{-1}A_{lm}d_m\| \tag{3.9}$$

holds for all $l \in S$. Since $J_P^+(D^{-1}AD) = J_P^+(A) = S$ and (3.9) shows $D^{-1}\widehat{A}D \in IBSD_s$, Theorem 3.4 yields that for any eigenvalue $\lambda$ of $D^{-1}AD$, we have $Re(\lambda) > 0$. Again, since $A$ has the same eigenvalues as $D^{-1}AD$, for any eigenvalue $\lambda$ of $A$, we have $Re(\lambda) > 0$. This completes the proof. □

Using Lemma 2.2 and Theorem 3.5, we can obtain the following corollary.

COROLLARY 3.6. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) = S$. If $\widehat{A} \in IBID_s$, where $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ is defined in* (3.6)*, then for any eigenvalue $\lambda$ of $A$, we have $Re(\lambda) > 0$.*

The following will extend the result of Theorem 1.1 to II$-$block diagonally dominant matrices and II$-$block $H-$matrices. First, we will introduce some relevant lemmas.

LEMMA 3.7. (see [3]) *Let $A \in C^{n\times n}$. Then the following conclusions are equivalent.*

1. *$A$ is Hermitian positive definite;*
2. *$A$ is Hermitian and each eigenvalue of $A$ is positive;*
3. *$A^{-1}$ is also Hermitian positive definite.*

LEMMA 3.8. *Let $A \in C^{n\times n}$ be nonsingular. Then*

$$\|A^{-1}\|^{-1} = \tau(A^HA), \tag{3.10}$$

*where $\tau(A^HA)$ denote the minimal eigenvalue of the matrix $A^HA$.*

*Proof.* It follows from equality (2.8) in Lemma 2.5 that

$$\|A^{-1}\| = \rho[(AA^H)^{-1}] = \rho[(A^HA)^{-1}] = \frac{1}{\tau(A^HA)}, \tag{3.11}$$

which yields equality (3.10). □

LEMMA 3.9. *Let $A \in C^{n\times n}$ be Hermitian positive definite and let $B \in C^{n\times m}$. Then for arbitrary complex number $\alpha \neq 0$ with $Re(\alpha) \geq 0$, we have*

$$\|(\alpha I + A)^{-1}B\| \leq \|A^{-1}B\|, \tag{3.12}$$

*where $I$ is identity matrix and $\|A\|$ is the spectral norm of the matrix $A$.*

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*Proof.* The theorem is obvious if $B = 0$. Since $A$ is Hermitian positive definite and $\alpha \neq 0$ with $Re(\alpha) \geq 0$, $\alpha I + A$ is non-Hermitian positive definite. Hence, $\alpha I + A$ is nonsingular. As a result, $(\alpha I + A)^{-1}B \neq 0$ and consequently $\|(\alpha I + A)^{-1}B\| \neq 0$ for $B \neq 0$. Thus, if $B \neq 0$, it follows from (2.1) and (2.2) that for arbitrary vector $x,\ y \in C^m,\ \|x\| = \|y\| = 1$, we have

$$
\begin{aligned}
\frac{\|A^{-1}B\|}{\|(\alpha I + A)^{-1}B\|} &= \frac{\sup\limits_{\|x\|=1} (\|A^{-1}Bx\|)}{\sup\limits_{\|y\|=1} (\|(\alpha I + A)^{-1}By\|)} \\
&\geq \frac{\|A^{-1}By\|}{\sup\limits_{\|y\|=1} (\|(\alpha I + A)^{-1}By\|)} \quad (set\ x = y) \\
&= \inf_{\|y\|=1} \left( \frac{\|A^{-1}By\|}{\|(\alpha I + A)^{-1}By\|} \right) \\
&= \inf_{0 \neq z \in C^n} \left( \frac{\|A^{-1}z\|}{\|(\alpha I + A)^{-1}z\|} \right) \quad (set\ z = By \in C^n) \\
&= \inf_{0 \neq u \in C^n} \left( \frac{\|A^{-1}(\alpha I + A)u\|}{\|u\|} \right) \quad (u = (\alpha I + A)^{-1}z \in C^n) \\
&= \|[A^{-1}(\alpha I + A)]^{-1}\|^{-1} \quad (from (2.2)) \\
&= \|(\alpha A^{-1} + I)^{-1}\|^{-1}.
\end{aligned} \tag{3.13}
$$

According to Lemma 3.8,

$$
\begin{aligned}
\|(\alpha A^{-1} + I)^{-1}\|^{-1} &= \tau[(\alpha A^{-1} + I)^H(\alpha A^{-1} + I)] \\
&= \tau(I + 2Re(\alpha)A^{-1} + |\alpha|^2 A^{-2}).
\end{aligned} \tag{3.14}
$$

Since $A$ is Hermitian positive definite, it follows from Lemma 3.7 that $A^{-1}$ and $A^{-2}$ are also. Therefore, $\alpha \neq 0$, together with $Re(\alpha) \geq 0$, implies that $2Re(\alpha)A^{-1} + |\alpha|^2 A^{-2}$ is also Hermitian positive definite. As a result, $\tau(2Re(\alpha)A^{-1} + |\alpha|^2 A^{-2}) > 0$. Thus, following (3.13) and (3.14), we get

$$
\begin{aligned}
\frac{\|A^{-1}B\|}{\|(\alpha I + A)^{-1}B\|} &\geq \|(\alpha A^{-1} + I)^{-1}\|^{-1} \\
&= \tau(I + 2Re(\alpha)A^{-1} + |\alpha|^2 A^{-2}) \\
&= 1 + \tau(2Re(\alpha)A^{-1} + |\alpha|^2 A^{-2}) \\
&> 1,
\end{aligned} \tag{3.15}
$$

from which it is easy to obtain (3.12). This completes the proof. ◻

THEOREM 3.10. *Let* $A = (A_{lm})_{s\times s} \in \text{II}BSD_s \cup \text{II}BID_s$ *with* $J_P^+(A) = S$. *If* $\widehat{A} = A$, *where* $\widehat{A}$ *is defined in* (3.6)*, then for any eigenvalue* $\lambda$ *of* $A$*, we have* $Re(\lambda) > 0$.

*Proof.* We prove by contradiction. Assume that $\lambda$ is any eigenvalue of $A$ with $Re(\lambda) \leq 0$. Then the matrix $\lambda I - A$ is singular. Since $\widehat{A} = A$ and (3.6) imply that the

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diagonal block $A_{ll}$ of $A$ is Hermitian for all $l \in S$, $J_P^+(A) = S$ yields that the diagonal block $A_{ll}$ of $A$ is Hermitian positive definite for all $l \in S$. Thus $\lambda I - A_{ll}$ is non-Hermitian negative definite for all $l \in S$. As a result, $D(\lambda) = diag(\lambda I - A_{11}, \cdots, \lambda I - A_{ss})$ is also non-Hermitian negative definite. Therefore, the matrix

$$\mathscr{A}(\lambda) := [D(\lambda)]^{-1}(\lambda I - A) = (\mathscr{A}_{lm})_{s\times s} \in C_s^{n\times n}$$

is singular, where $\mathscr{A}_{ll} = I_l$, the $n_l \times n_l$ identity matrix and $\mathscr{A}_{lm} = (\lambda I - A_{ll})^{-1} A_{lm}$ for $l \neq m$ and $l, m \in S$. It follows from Lemma 3.9 that

$$\|(\lambda I - A_{ll})^{-1} A_{lm}\| \leq \|A_{ll}^{-1} A_{lm}\| \tag{3.16}$$

for $l \neq m$ and $l, m \in S$. Since $A \in \text{II}BSD_s \cup \text{II}BID_s$,

$$1 \geq \sum_{m=1, m\neq l}^{s} \|A_{ll}^{-1} A_{lm}\| \tag{3.17}$$

holds for all $l \in S$ and the inequality in (3.17) holds strictly for at least one $i \in S$. Then from (3.16) and (3.17), we have

$$1 \geq \sum_{m=1, m\neq l}^{s} \|(\lambda I - A_{ll})^{-1} A_{lm}\| \tag{3.18}$$

holds for all $l \in S$ and the inequality in (3.18) holds strictly for at least one $i \in S$. If $A \in \text{II}BSD_s$, then the inequality in (3.17) and hence the one in (3.18) both hold strictly for all $i \in S$. That is, $\mathscr{A}(\lambda) \in IBSD_s$. If $A$ is block irreducible, then so is $\mathscr{A}(\lambda)$. As a result, $A \in \text{II}BID_s$ yields $\mathscr{A}(\lambda) \in IBID_s$. Therefore, $A \in \text{II}BSD_s \cup \text{II}BID_s$ which implies $\mathscr{A}(\lambda) \in IBSD_s \cup IBID_s$. Using Lemma 2.1, $\mathscr{A}(\lambda)$ is nonsingular and consequently $\lambda I - A$ is nonsingular, which contradicts the singularity of $\lambda I - A$. This shows that the assumption is incorrect. Thus, for any eigenvalue $\lambda$ of $A$, we have $Re(\lambda) > 0$. □

THEOREM 3.11. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) = S$. If $\widehat{A} \in \text{II}BSD_s \cup \text{II}BID_s$, where $\widehat{A}$ is defined by* (3.6)*, then for any eigenvalue $\lambda$ of $A$, we have $Re(\lambda) >$ 0.*

*Proof.* Since $\widehat{A} \in \text{II}BSD_s \cup \text{II}BID_s$, it follows from Lemma 2.2 that $\widehat{A} \in \text{II}BH_s$ and $\widehat{A}^H \in \text{II}BH_s$. Following Lemma 2.3, there exists a block diagonal matrix $D = diag(d_1 I_1, \cdots, d_s I_s)$, where $d_l > 0$, $I_l$ is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum_{l=1}^{s} n_l = n$, such that $\widehat{A}^H D = (D\widehat{A})^H \in \text{II}BSD_s$. Again, since $\widehat{A}$ is II$-$block strictly or irreducibly diagonally dominant by row, so is $D\widehat{A}$. Furthermore, (3.6) implies that diagonal blocks of $\widehat{A}$ are all Hermitian, and hence, so are the diagonal blocks of $D\widehat{A}$ and $(D\widehat{A})^H$. Therefore, from the definition of II$-$block strictly or irreducibly

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diagonally dominant matrix by row, we have

$$1 \geq \sum_{m=1, m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_l\widehat{A}_{lm})\| = \sum_{m=1, m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_lA_{lm})\| \tag{3.19}$$

and

$$1 > \sum_{m=1, m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_m\widehat{A}_{ml}^H)\| = \sum_{m=1, m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_mA_{ml}^H)\| \tag{3.20}$$

hold for all $l \in S$. Therefore, according to (3.19) and (3.20), we have

$$\begin{aligned} & 1 - \sum_{m=1, m\neq l}^{s} \|(d_lA_{ll} + d_lA_{ll}^H)^{-1}(d_lA_{lm} + d_mA_{ml}^H)\| \\ &= 1 - \sum_{m=1, m\neq l}^{s} \|(2d_l\widehat{A}_{ll})^{-1}(d_lA_{lm} + d_mA_{ml}^H)\| \\ &\geq 1 - \frac{1}{2}\sum_{m=1, m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_lA_{ml}) + (d_l\widehat{A}_{ll})^{-1}(d_mA_{ml}^H)\| \\ &\geq 1 - \frac{1}{2}\sum_{m=1, m\neq l}^{s} \left[\|(d_l\widehat{A}_{ll})^{-1}(d_lA_{ml}\| + \|d_l\widehat{A}_{ll})^{-1}(d_mA_{ml}^H)\|\right] \\ &\geq \frac{1}{2}\left[(1 - \sum_{m=1, m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_lA_{ml}\|) + (1 - \sum_{m=1, m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_mA_{lm}^H\|)\right] \\ &> 0, \end{aligned}$$

which indicates that $DA+(DA)^H = D\widehat{A}+(D\widehat{A})^H \in \mathrm{II}BSD_s$. Again, since $J_P^+(A) = S$, the diagonal block of $A+A^H$, $A_{ll}+A_{ll}^H = 2\widehat{A}_{ll}$ is Hermitian positive definite for all $l \in S$. As a result, the diagonal block of $DA+(DA)^H$, $d_lA_{ll}+d_lA_{ll}^H = 2d_l\widehat{A}_{ll}$ is also Hermitian positive definite for all $l \in S$. Thus, it follows from Theorem 3.10 that for any eigenvalue $\mu$ of $DA+(DA)^H$, $Re(\mu) > 0$. Since $DA+(DA)^H$ is Hermitian, each eigenvalue $\mu$ of $DA+(DA)^H$ is positive. Then Lemma 3.7 yields that $DA+(DA)^H$ is Hermitian positive definite. From the proof above, we conclude that there exists a Hermitian positive definite matrix $D = diag(d_1I_1, \cdots, d_sI_s)$, where $d_l > 0$, $I_l$ is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum_{l=1}^{s} n_l = n$, such that $DA+(DA)^H$ is Hermitian positive definite. It follows from Lyapunov's theorem (see [4], pp.96) that $A$ is positive stable, i.e., for any eigenvalue $\lambda$ of $A$, we have $Re(\lambda) > 0$, which completes the proof. ▯

THEOREM 3.12. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) = S$. If $\widehat{A} \in \mathrm{II}BH_s$, where $\widehat{A}$ is defined in* (3.6)*, then for any eigenvalue $\lambda$ of $A$, we have $Re(\lambda) > 0$.*

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*Proof.* Similar to the proof of Theorem 3.5, we can obtain the proof of this theorem by Lemma 2.3 and Theorem 3.11. □

Using Theorem 3.5 and Theorem 3.12, we obtain a sufficient condition for the system (1.1) to be globally stable.

COROLLARY 3.13. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(-A) = S$. If $\widehat{A} \in \text{II}BH_s$ $(IBH_s)$, where $\widehat{A}$ is defined in* (3.6)*, then the equilibrium $\hat{x}$ of system* (1.1) *is globally stable.*

**4. The eigenvalue distribution of block diagonally dominant matrices and block $H-$matrices.** In this section, some theorems on the eigenvalue distribution of block diagonally dominant matrices and block $H-$matrices are presented, generalizing Theorem 1.2, Theorem 1.3 and Theorem 1.4.

THEOREM 4.1. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) \cup J_P^-(A) = S$. If $\widehat{A} \in IBSD_s$, where $\widehat{A}$ is defined in* (3.6)*, then $A$ has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part.*

*Proof.* Suppose every block Gersgorin disk of the matrix $A$ given in (3.5)

$$\Gamma_l : \|(A_{ll} - \lambda I)^{-1}\|^{-1} \le \sum_{m=1, m\ne l}^{s} \|A_{lm}\|, \quad l \in S.$$

Let

$$R_1 = \bigcup_{k\in J_P^+(A)} \Gamma_k, \quad R_2 = \bigcup_{k\in J_P^-(A)} \Gamma_k.$$

Since $J_P^+(A) \cup J_P^-(A) = S$, then $R_1 \cup R_2 = \bigcup_{l\in S} \Gamma_l$. Therefore, it follows from Lemma 3.3 that each eigenvalue $\lambda$ of the matrix $A$ lies in $R_1 \cup R_2$. Furthermore, $R_1$ lies on the right of imaginary axis, $R_2$ lies on the left of the imaginary axis in the imaginary coordinate plane. Then $A$ has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part in $R_1$, and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part in $R_2$. Otherwise, $A$ has an eigenvalue $\lambda_{k_0} \in R_1$ with $Re(\lambda_{k_0}) \le 0$ such that for at least one $l \in J_P^+(A)$,

$$\|(A_{ll} - \lambda_0 I)^{-1}\|^{-1} \le \sum_{m=1, m\ne l}^{s} \|A_{lm}\|. \tag{4.1}$$

Then it follows from Lemma 3.2 that

$$\|H_{ll}^{-1}\|^{-1} \le \|(A_{ll} - \lambda_{k_0} I)^{-1}\|^{-1} \tag{4.2}$$

for at least one $l \in J_P^+(A)$. It then follows from (4.1) and (4.2) that

$$\|H_{ll}^{-1}\|^{-1} \le \sum_{m=1, m\ne l}^{s} \|A_{lm}\| \tag{4.3}$$

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for at least one $l \in J_P^+(A)$. Inequality (4.3) contradicts $\widehat{A} \in IBSD_s$. By the same method, we can obtain the same result if there exists an eigenvalue with nonnegative real part in $R_2$. Hence, if $\widehat{A} \in IBSD_s$ and $J_P^+(A) \cup J_P^-(A) = S$, then for an arbitrary eigenvalue $\lambda_i$ of $A$, we have $Re(\lambda_i) \neq 0$ for $i = 1, 2, \cdots, n$. Again, $J_P^+(A) \cap J_P^-(A) = \emptyset$ yields $R_1 \cap R_2 = \emptyset$. Since $R_1$ and $R_2$ are both closed sets and $R_1 \cap R_2 = \emptyset$, $\lambda_i$ in $R_1$ can not jump into $R_2$ and $\lambda_i \in R_2$ can not jump into $R_1$. Thus, $A$ has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part in $R_1$ and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part in $R_2$. This completes the proof. □

THEOREM 4.2. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) \cup J_P^-(A) = S$. If $\widehat{A} \in IBH_s$, where $\widehat{A}$ is defined in* (3.6)*, then $A$ has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part.*

*Proof.* Since $\widehat{A} \in IBH_s$, it follows from Lemma 2.3 and the proof of Theorem 3.5 that there exists a block diagonal matrix $D = diag(d_1 I_1, \cdots, d_s I_s)$, where $d_l > 0$, $I_l$ is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum_{l=1}^s n_l = n$, such that $D^{-1}\widehat{A}D \in IBSD_s$, i.e.,

$$\|H_{ll}^{-1}\|^{-1} > \sum_{m=1, m\neq l}^{s} d_l^{-1}\|A_{lm}\| d_m = \sum_{m=1, m\neq l}^{s} \|d_l^{-1} A_{lm} d_m\| \tag{4.4}$$

holds for all $l \in S$. Since $J_P^+(D^{-1}AD) = J_P^+(A)$ and $J_P^-(D^{-1}AD) = J_P^-(A)$, Theorem 3.4 yields that $D^{-1}AD$ has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part. Again, since $A$ has the same eigenvalues as $D^{-1}AD$, $A$ has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part. This completes the proof. □

Following Lemma 2.2 and Theorem 4.2, we have the following corollary.

COROLLARY 4.3. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) \cup J_P^-(A) = S$. If $\widehat{A} \in IBID_s$, where $\widehat{A}$ is defined in* (3.6)*, then $A$ has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part.*

Now, we consider the eigenvalue distribution of II−block diagonally dominant matrices and II−block $H-$matrices. In the following lemma, a further extension of the *Gersgorin Circle Theorem* is given.

LEMMA 4.4. *If for each block row of the block matrix $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$, there exists at least one off-diagonal block not equal to zero, then for each eigenvalue $\lambda$ of $A$,*

$$\sum_{m=1, m\neq l}^{s} \|(A_{ll} - \lambda I)^{-1} A_{lm}\| \geq 1 \tag{4.5}$$

*holds for at least one $l \in S$, where $\|(A_{ll} - \lambda I)^{-1} A_{lm}\| \to \infty$ (defined in* (2.4)*) if $A_{ll} - \lambda I$*

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*is singular and* $A_{lm} \neq 0$ *for* $l \neq m$ *and* $l, m \in S$.

*Proof.* The proof is by contradiction. Assume that $\lambda$ is an arbitrary eigenvalue of $A$ such that

$$\sum_{m=1, m\neq l}^{s} \|(A_{ll} - \lambda I)^{-1} A_{lm}\| < 1 \tag{4.6}$$

holds for all $l \in S$. It follows from (4.6) that $A_{ll} - \lambda I$ is nonsingular for all $l \in S$. Otherwise, there exists at leas one $l_0 \in S$ such that $A_{l_0 l_0} - \lambda I$ is singular. Since there exists at least one off-diagonal block $A_{l_0 m} \neq 0$ for $m \in S$ in the $l_0$th block row, (2.4) yields $\|(A_{l_0 l_0} - \lambda I)^{-1} A_{l_0 m}\| \to \infty$ and consequently, $\sum_{m=1,m\neq l}^{s} \|(A_{l_0 l_0} - \lambda I)^{-1} A_{l_0 m}\| \to \infty$, which contradicts (4.6). Therefore, $A_{ll} - \lambda I$ is nonsingular for all $l \in S$, which leads to the nonsingularity of the block diagonal matrix $D(\lambda) = diag(A_{11} - \lambda I, \cdots, A_{ss} - \lambda I)$. Further, (4.6) also shows $A - \lambda I \in \mathrm{II}BSD_s$ Thus, $\mathscr{A}(\lambda) := [D(\lambda)]^{-1}(A - \lambda I) = (\mathscr{A}_{lm})_{s\times s} \in IBSD_s$. Then, we have from Lemma 2.1 that $\mathscr{A}(\lambda)$ is nonsingular. As a result, $A - \lambda I$ is nonsingular, which contradicts the assumption that $\lambda$ is an arbitrary eigenvalue of $A$. Hence, if $\lambda$ is an arbitrary eigenvalue of $A$, then $A - \lambda I$ cannot be II$-$block diagonally dominant, which gives the conclusion of this lemma. ☐

Theorem 4.5. *Let* $A = (A_{lm})_{s\times s} \in \mathrm{II}BSD_s$ *with* $J_P^+(A) \cup J_P^-(A) = S$. *If* $\widehat{A} = A$, *where* $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ *is defined in* (3.6)*, then* $A$ *has* $\sum_{k\in J_P^+(A)} n_k$ *eigenvalues with positive real part and* $\sum_{k\in J_P^-(A)} n_k$ *eigenvalues with negative real part.*

*Proof.* The proof proceeds with the following two cases.

(i) If for each block row of the block matrix $A$, there exists at least one off-diagonal block not equal to zero, one may suppose that every block Gersgorin disk of the matrix $A$ given in (4.5) is

$$G_l : 1 \leq \sum_{m=1,m\neq l}^{s} \|(A_{ll} - \lambda I)^{-1} A_{lm}\|, \quad l \in S.$$

Let

$$\widetilde{R}_1 = \bigcup_{k\in J_P^+(A)} G_k, \quad \widetilde{R}_2 = \bigcup_{k\in J_P^-(A)} G_k.$$

Since $J_P^+(A) \cup J_P^-(A) = S$, then $\widetilde{R}_1 \cup \widetilde{R}_2 = \bigcup\limits_{l\in S} G_l$. Therefore, it follows from Lemma 4.4 that each eigenvalue $\lambda$ of the matrix $A$ lies in $\widetilde{R}_1 \cup \widetilde{R}_2$. Furthermore, $\widetilde{R}_1$ lies on the right of imaginary axis, $\widetilde{R}_2$ lies on the left of the imaginary axis in the imaginary coordinate plane. Then $A$ has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part in $\widetilde{R}_1$, and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part in $\widetilde{R}_2$. Otherwise, assume

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that $A$ has an eigenvalue $\lambda_{k_0} \in \widetilde{R}_1$ such that $Re(\lambda_{k_0}) \leq 0$. Therefore, we have from Lemma 4.4 that

$$1 \leq \sum_{m=1, m\neq l}^{s} \|(A_{ll} - \lambda_0 I)^{-1} A_{lm}\| \tag{4.7}$$

holds for at least $l \in J_P^+(A)$. Since (3.6) and $\widehat{A} = A$ imply that the diagonal blocks of $A$ are all Hermitian, the diagonal block $A_{ll}$ of the block matrix $A$ is Hermitian positive definite for all $l \in J_P^+(A)$. Then it follows from Lemma 3.9 that

$$\|(\lambda I - A_{ll})^{-1} A_{lm}\| \leq \|A_{ll}^{-1} A_{lm}\| \tag{4.8}$$

for all $l \in J_P^+(A)$ and $m \neq l$, $m \in S$. Inequalities (4.7) and (4.8) yield that

$$1 \leq \sum_{m=1, m\neq l}^{s} \|A_{ll}^{-1} A_{lm}\| \tag{4.9}$$

holds for at least $l \in J_P^+(A)$. Inequality (4.9) contradicts $A \in \mathrm{II}BSD_s$. In the same method, we can obtain the same result if there exists an eigenvalue with nonnegative real part in $\widetilde{R}_2$. Hence, if $A \in \mathrm{II}BSD_s$ and $J_P^+(A) \cup J_P^-(A) = S$, then for arbitrary eigenvalue $\lambda_i$ of $A$, we have $Re(\lambda_i) \neq 0$ for $i = 1, 2, \cdots, n$. Again, $J_P^+(A) \cap J_P^-(A) = \emptyset$ yields $\widetilde{R}_1 \cap \widetilde{R}_2 = \emptyset$. Since $\widetilde{R}_1$ and $\widetilde{R}_2$ are all closed set and $\widetilde{R}_1 \cap \widetilde{R}_2 = \emptyset$, $\lambda_i$ in $\widetilde{R}_1$ can not jump into $\widetilde{R}_2$ and $\lambda_i \in \widetilde{R}_2$ can not jump into $\widetilde{R}_1$. Thus, $A$ has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part in $\widetilde{R}_1$ and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part in $\widetilde{R}_2$. This completes the proof of (i).

(ii) The following will prove the case when there exist some block rows of the block matrix $A$ with all their off-diagonal blocks equal to zero. Let $\omega \subseteq S$ denote the set containing block row indices of such block rows. Then there exists an $n \times n$ permutation matrix $P$ such that

$$PAP^T = \begin{bmatrix} A(\omega') & A(\omega', \omega) \\ 0 & A(\omega) \end{bmatrix}, \tag{4.10}$$

where $\omega' = S - \omega$, $A(\omega') = (A_{lm})_{l,m\in\omega'}$ has no block rows with all their off-diagonal blocks equal to zero, $A(\omega) = (A_{lm})_{l,m\in\omega}$ is a block diagonal matrix and $A(\omega', \omega) = (A_{lm})_{l\in\omega', m\in\omega}$. It is easy to see that the partition of (4.10) does not destroy the partition of (2.6). Further, (4.10) shows that

$$\sigma(A) = \sigma(A(\omega')) \cup \sigma(A(\omega)), \tag{4.11}$$

where $\sigma(A)$ denotes the spectrum of the matrix $A$. Since and $A(\omega')$ is a block principal submatrix of $A$ and $J_P^+(A) \cup J_P^-(A) = S$, $J_P^+[A(\omega')] \cup J_P^-[A(\omega')] = \omega'$. Further,

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$A = \widehat{A} \in \mathrm{II}BSD_s$ gives $A(\omega') = \widehat{A}(\omega') \in \mathrm{II}BSD_{|\omega'|}$. Again, since $A(\omega') = (A_{lm})_{l,m\in\omega'}$ has no block rows with all their off-diagonal blocks equal to zero, i.e., for each block row of the block matrix $A(\omega')$, there exists at least one off-diagonal block not equal to zero, it follows from the proof of (i) that $A(\omega')$ has $\sum_{k\in J_P^+[A(\omega')]} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-[A(\omega')]} n_k$ eigenvalues with negative real part.
Let's consider the matrix $A(\omega)$. $A(\omega)$ being a block principal submatrix of the block matrix $A = (A_{lm})_{s\times s} \in \mathrm{II}BSD_s$ with $J_P^+(A) \cup J_P^-(A) = S$ gives $J_P^+[A(\omega)] \cup J_P^-[A(\omega)] = \omega$. Since $A(\omega) = (A_{lm})_{l,m\in\omega}$ is a block diagonal matrix, the diagonal block $A_{ll}$ of $A(\omega)$ is either non-Hermitian positive definite or non-Hermitian negative definite for each $l \in \omega$, and consequently

$$(4.12)\quad \sigma(A(\omega)) = \bigcup_{l\in\omega} \sigma(A_{ll}) = \left( \bigcup_{l\in J_P^+[A(\omega)]} \sigma(A_{ll}) \right) \bigcup \left( \bigcup_{l\in J_P^-[A(\omega)]} \sigma(A_{ll}) \right).$$

The equality (4.12) and Lemma 2.4 shows that $A(\omega)$ has $\sum_{k\in J_P^+[A(\omega)]} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-[A(\omega)]} n_k$ eigenvalues with negative real part.
Since

$$\begin{aligned} J_P^+(A) \cup J_P^-(A) &= S = \omega \cup \omega' \\ &= \left(J_P^+[A(\omega)] \cup J_P^-[A(\omega)]\right) \cup \left(J_P^+[A(\omega')] \cup J_P^-[A(\omega')]\right) \ , \\ &= \left(J_P^+[A(\omega)] \cup J_P^+[A(\omega')]\right) \cup \left(J_P^-[A(\omega) \cup J_P^-[A(\omega')]\right) \end{aligned}$$

we have

$$(4.13)\qquad J_P^+(A) = J_P^+[A(\omega)] \cup J_P^+[A(\omega')], \quad J_P^-(A) = J_P^-[A(\omega)] \cup J_P^-[A(\omega')].$$

Again, $\omega' = S - \omega$ implies $\omega \cap \omega' = \emptyset$, which yields

$$(4.14)\qquad J_P^+[A(\omega)] \cap J_P^+[A(\omega')] = \emptyset, \quad J_P^-[A(\omega)] \cap J_P^-[A(\omega')] = \emptyset.$$

According to (4.13), (4.14) and the partition (2.6) of $A$, it is not difficult to see that

$$(4.15)\qquad \begin{aligned} \textstyle\sum_{k\in J_P^+(A)} n_k &= \textstyle\sum_{k\in J_P^+[A(\omega)]} n_k + \sum_{k\in J_P^+[A(\omega')]} n_k, \\ \textstyle\sum_{k\in J_P^-(A)} n_k &= \textstyle\sum_{k\in J_P^-[A(\omega)]} n_k + \sum_{k\in J_P^-[A(\omega')]} n_k. \end{aligned}$$

From (4.15) and the eigenvalue distribution of $A(\omega')$ and $A(\omega)$ given above, it is not difficult to see that $A$ has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part. We conclude from the proof of (i) and (ii) that the proof of this theorem is completed. □

THEOREM 4.6. *Let $A = (A_{lm})_{s\times s} \in \mathrm{II}BH_s$ with $J_P^+(A) \cup J_P^-(A) = S$. If $\widehat{A} = A$, where $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ is defined in* (3.6)*, then $A$ has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part.*

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*Proof.* Since $\widehat{A} = A \in \amalg BH_s$, it follows from Lemma 2.3 that there exists a block diagonal matrix $D = diag(d_1 I_{n_1}, d_2 I_{n_2}, \cdots, d_s I_{n_s})$, where $d_l > 0$, $I_{n_l}$ is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum\limits_{l=1}^{s} n_l = n$, such that $\widehat{A}D = AD \in \amalg BSD_s$, i.e.,

$$(4.16)\ 1 > \sum_{m=1, m\neq l}^{s} \|(A_{ll}d_l)^{-1}(A_{lm}d_m)\| = \sum_{m=1, m\neq l}^{s} \|(d_l^{-1}A_{ll}d_l)^{-1}(d_l^{-1}A_{lm}d_m)\|$$

holds for all $l \in S$. Inequality (4.16) shows $B = D^{-1}AD = D^{-1}\widehat{A}D = \widehat{B} \in \amalg BSD_s$. Since $B = D^{-1}AD$ has the same diagonal blocks as the matrix $A$, it follows from Theorem 4.5 that $B$ has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part, so does $A$. ◻

Corollary 4.7. *Let $A = (A_{lm})_{s\times s} \in \amalg BID_s$ with $J_P^+(A) \cup J_P^-(A) = S$. If $\widehat{A} = A$, where $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ is defined in (3.6), then $A$ has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part.*

*Proof.* The proof is obtain directly by Lemma 2.2 and Theorem 4.6. ◻

**5. Conclusions.** This paper concerns the eigenvalue distribution of block diagonally dominant matrices and block block $H-$matrices. Following the result of Feingold and Varga, a well-known theorem of Taussky on the eigenvalue distribution is extended to block diagonally dominant matrices and block $H-$matrices with each diagonal block being non-Hermitian positive definite. Then, the eigenvalue distribution of some special matrices including block diagonally dominant matrices and block $H$-matrices is studied further to give the conditions on the block matrix $A$ such that the matrix $A$ has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part.

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