The numerical range of linear operators on the 2-dimensional Krein space

Hiroshi Nakazato
Natalia Bebiano
Joao da Providencia

Follow this and additional works at: http://repository.uwyo.edu/ela

Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1447
THE NUMERICAL RANGE OF LINEAR OPERATORS
ON THE 2-DIMENTIONAL KREIN SPACE*

HIROSHI NAKAZATO†, NATÁLIA BEBIANO‡, AND JOÃO DA PROVIDÊNCIA§

Abstract. The aim of this note is to provide the complete characterization of the numerical
range of linear operators on the 2-dimensional Krein space $\mathbb{C}^2$.

Key words. Krein space, Numerical range, Rank one operator.

AMS subject classifications. 46C20, 47A12.

1. Introduction. The concept of numerical range of linear operators on a
Hilbert space was introduced by Toeplitz [16] and has been generalized in several
directions. The theory of numerical ranges of linear operators on a Krein space has
also been considered by some authors (see [2, 3, 10, 11, 13, 14, 15] and the references
therein). There are many motivations for the study of the numerical range of lin-
ear operators on Hilbert spaces or Krein spaces. We enumerate some of them: the
localization of the spectrum of an operator, related inequalities, control theory and
applications to physics (cf. [8]). Recently, an application of a generalized numer-
cal range to NMR spectroscopy has been discussed (cf. [7, 12]). The aim of this
paper is the complete determination of the numerical range of linear operators on
the 2-dimensional Krein space $\mathbb{C}^2$. By addition of scalar operators, the study of the
numerical range of operators on $\mathbb{C}^2$ is reduced to that of rank one operators. The
numerical range of such rank one operators has been already investigated in [14] for
non-neutral vectors.

Let $C, A$ be (non-zero) rank one operators on the 2-dimensional Krein space $\mathbb{C}^2$,
endowed with the indefinite inner product space $[\cdot, \cdot]$ defined by $[\xi, \nu] = (J\xi, \nu) = \nu^* J\xi$
for $J = I_1 \oplus (-I_1)$. We refer [1, 4, 6] for general reference on Krein spaces or
Krein spaces operators. For the rank one operators $C, A$, there exist non-zero vectors

Received by the editors on November 22, 2010. Accepted for publication on April 2, 2011.
Handling Editor: Michael Tsatsomeros.

†Department of Mathematical Sciences, Hirosaki University, 036-8561, Japan (nakahr@cc.hirosaki-u.ac.jp).
‡Mathematics Department, University of Coimbra, P3001-454 Coimbra, Portugal (bebiano@mat.uc.pt).
§Physics Department, University of Coimbra, P3004-516 Coimbra, Portugal (providencia@teor.fis.uc.pt).

430
The Numerical Range of Linear Operators on the 2-Dimensional Krein Space

\( \eta, \zeta, \kappa, \tau \) such that

\[
C\xi = [\xi, \eta]_{C}, \quad A\xi = [\xi, \kappa]_{C} \quad \xi \in \mathbb{C}^{2}.
\]

Denote by \( SU(1, 1) \) the group of \( 2 \times 2 \) complex matrices \( U \) with determinant 1 such that \( U^*JU = J \). We consider the indefinite \( C \)-numerical range of \( A \) denoted and defined by

\[
W^J_C(A) = \{ [U\zeta, \kappa]_{C}[U\eta, \tau] : U \in SU(1, 1) \},
\]

which has been characterized in [14] for \( \eta, \zeta, \kappa, \tau \), non-neutral vectors, that is, \( [\eta, \eta], [\zeta, \zeta], [\kappa, \kappa], [\tau, \tau] \) do not vanish. An analogous object for a 2-dimensional Hilbert space is the \( C \)-numerical range of \( A \) defined as

\[
W_C(A) = \{ \kappa^*U\zeta \tau^*U\eta : U \in SU(2) \},
\]

for rank one operators \( C, A \). The range \( W_C(A) \) is a (possible degenerate) closed elliptical disc (cf. [9]). This paper treats the analogous object for Krein spaces.

The main aim of this note is to complete the characterization of \( W^J_C(A) \) considering the case of the vectors \( \eta, \zeta, \kappa, \tau \) being neutral. For the range \( W_C(A) \) of any dimensional matrices \( A, C \), the numerical method to draw the boundary is given in [7] based on a result in [5]. A numerical algorithm is not known for Krein spaces numerical ranges except for some special cases. We give a complete characterization for operators on 2-dimensional spaces.

The following classification takes place, being the different cases treated in the next five sections.

**First Case:** All the vectors \( \eta, \zeta, \kappa, \tau \) are neutral.

**Second Case:** One of the vectors \( \eta, \zeta, \kappa, \tau \) is non-neutral and the other three are neutral.

**Third Case:** The vectors \( \kappa, \tau \) are neutral and \( \eta, \zeta \) are non-neutral.

**Fourth Case:** The vectors \( \kappa, \zeta \) are neutral and \( \eta, \tau \) are non-neutral.

**Fifth Case:** The vectors \( \kappa, \eta \) are neutral and \( \zeta, \tau \) are non-neutral.

**Sixth Case:** One of the vectors \( \eta, \zeta, \kappa, \tau \) is neutral and the other three are non-neutral.

Using the concrete description of \( W^J_C(A) \) given in Sections 2-6 and in [14], we prove the following results in Section 7.

**Theorem 1.1.** Let \( C, A \) be arbitrary linear operators on a 2-dimensional Krein space, and \( J = I_1 \oplus -I_1 \). If the boundary of \( W^J_C(A) \) is non-empty, then it is a singleton or it lies on a possibly degenerate conic.

**Theorem 1.2.** Let \( C, A \) be arbitrary linear operators on the Krein space \( \mathbb{C}^2 \), and \( J = I_1 \oplus -I_1 \). Then the fundamental group \( \pi_1(W^J_C(A)) \) of \( W^J_C(A) \) is a trivial group,
2. The first case. In the sequel, we identify the complex plane with $\mathbb{R}^2$ and we denote by $E_{ij}$ the $2 \times 2$ matrix with the $(i,j)$th entry equal to 1 and all the others 0. For $V, W \in U(1,1)$, we have

\[(2.1) \quad VAV^{-1}\xi = [V^{-1}\xi, \kappa]V\tau = [\xi, V\kappa]V\tau, \quad WCW^{-1}\xi = [W^{-1}\xi, \eta]\xi = [\xi, W\eta]W\zeta,\]

and so we may assume that $\zeta = (1,1)^T, \tau = (1, -1)^T$. We also may consider that $\kappa = (\overline{k_1}, -\overline{k_2}), \eta = (\overline{q_1}, \overline{q_2})$, with $|k_1| = |k_2| = |q_1| = |q_2| = 1$. Under these assumptions, we find

\[
W_C^J(A) = \{(k_1\alpha + k_1\overline{\beta} + k_2\overline{\alpha} + k_2\beta)(q_1\overline{\alpha} + q_2\beta + q_1\overline{\beta} + q_2\alpha) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}.
\]

Writing $k_1 = \exp(i\eta)\exp(i\theta), k_2 = \exp(i\eta)\exp(-i\theta), q_1 = \exp(it)\exp(i\phi), q_2 = \exp(it)\exp(-i\phi), s, t, \theta, \phi \in \mathbb{R}$, we obtain

\[
(2.2) \quad W_C^J(A) = \{4\exp(i\eta)\exp(i\theta)(\alpha + \overline{\beta})\exp(i\eta)\exp(i\phi)(\alpha + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}.
\]

Thus, $W_C^J(A)$ is contained in a straight line passing through the origin. We may assume that $\kappa = (\exp(-i\theta), -\exp(i\theta))^T, \eta = (\exp(-i\phi), \exp(i\phi))^T, \zeta = (1,1)^T$ and $\tau = (1, -1)^T$. Under these assumptions, we prove the following.

**Proposition 2.1.** Let $C = \exp(i\phi)(E_{11} + E_{22}) - \exp(-i\phi)(E_{12} + E_{22}), A = \exp(i\eta)(E_{11} - E_{22}) + \exp(-i\eta)(E_{12} - E_{22}), 0 \leq \theta, \phi \leq 2\pi$. If either $\theta$ or $\phi$ is not congruent to 0 modulo $\pi$, then $W_C^J(A)$ is the real line. Otherwise, $W_C^J(A) = [0, +\infty)$ for $\theta = \phi = 0$ or $\theta = \phi = \pi$, and $W_C^J(A) = (-\infty, 0]$ for $\theta = 0, \phi = \pi$ or $\theta = \pi, \phi = 0$.

**Proof.** Having in mind (2.2), we easily find for $0 \leq \theta, \phi \leq 2\pi$,

\[
W_C^J(A) = \{4\exp(i\theta)(\alpha + \overline{\beta})\exp(-i\phi)(\alpha + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}.
\]

For fixed $0 \leq u, v, \theta, \phi \leq 2\pi$, we study the behavior of the real valued function

\[
\psi(t) = 4(\cosh t \cos(u + \theta) + \sinh t \cos(v - \theta))\cosh t \cos(u - \phi) + \sinh t \cos(v - \phi) + \sinh t \cos(v - \phi))
\]

as $t \to +\infty$. For this purpose, we consider the sign of the function

\[
f(u, v) = (\cos(u + \theta) + \cos(v - \theta))\cos(u - \phi) + \cos(v - \phi))
\]

\[
= 4 \cos \left( \frac{u + v}{2} \right) \cos \left( \frac{u + v}{2} - \phi \right) \cos \left( \frac{u - v}{2} + \theta \right) \cos \left( \frac{u - v}{2} \right), \quad u, v \in \mathbb{R}.
\]
Hence, it is sufficient to determine the sign of the function $\tilde{f}(x, y) = 4\cos x \cos(x - \phi) \cos y \cos(y + \theta)$ for the variables ranging over the reals. We observe that function $\cos x \cos(x - \phi)$ takes positive and negative values except for $\phi \equiv 0 \bmod \pi$, while $\cos y \cos(y + \theta)$ takes positive and negative values except for $\theta \equiv 0 \bmod \pi$. Therefore, if either $\phi$ or $\theta$ is not congruent to 0 modulo $\pi$, then $f(u, v)$ takes positive and negative values, and so $W_C^f(A)$ coincides with the real line.

To finish the proof we consider the following cases: (1) $\theta = \phi = 0$; (2) $\theta = 0$ and $\phi = \pi$; (3) $\theta = \pi$ and $\phi = 0$; (4) $\theta = \phi = \pi$. We concentrate on the case (1). Considering $t = 0, u = \pi/2$, we produce the origin. Taking $u = v = 0$, we get $[0, +\infty) \subset W_C^f(A)$ and the reverse inclusion is clear.

The treatment of the remaining cases is similar and the proposition follows. □

3. The fourth and fifth cases. In the fourth case, we may consider $\zeta = (1, 1)^T$, $\kappa = (1, -1)$, $\tau = (1, 0)^T$, and $\eta = (1, 0)^T$ or $\eta = (0, 1)^T$. In the fifth case, we may take $\eta = (1, 1)^T$, $\kappa = (1, 1)^T$, $\zeta = (1, 0)^T$, and $\tau = (1, 0)^T$ or $\tau = (0, 1)^T$.

**Proposition 3.1.** If $C = E_{11} + E_{21}$ and $A = E_{11} + E_{12}$, then $W_C^f(A) = \mathbb{C} \setminus (\infty, 1] \cup \{0\}$.

**Proof.** Some computations yield

$$W_C^f(A) = \{(\alpha + \pi + \beta + \bar{\beta})\pi : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}$$

$$= \{2(\cosh t \cos \theta + \sinh t \cos \phi) \cosh t \exp(i\theta) : t \geq 0, 0 \leq \theta, \phi \leq 2\pi\}.$$ 

By choosing in (3.1) $\theta = \pi/2$ and $\phi = 0$ or $\phi = \pi$, we conclude that the imaginary axis is contained in $W_C^f(A)$. It can be easily seen that a non-zero real number $\lambda$ belongs to $W_C^f(A)$ if and only if it is of the form

$$\lambda(t, \phi) = 2(\cosh t + \sinh t \cos \phi) \cosh t, \ t \geq 0, \ 0 \leq \phi \leq 2\pi.$$ 

For a fixed $t$, $\lambda$ attains its minimum value when $\cos \phi = -1$ and this minimum equals $1 + \exp(-2\theta)$. Letting $t$ vary on its domain, we find $\lambda > 1$. Thus,

$$W_C^f(A) \cap \mathbb{R} = \{0\} \cup (1, +\infty).$$

Considering $0 < \theta < \pi$ and $\phi = \pi$ in (3.1), we find

$$\{2(\cosh t \cos \theta - \sinh t) \cosh t \exp(i\theta) : t \geq 0\} = \{\lambda \exp(i\theta) : \lambda \leq 2 \cos \theta\}$$

$$\subset W_C^f(A),$$

while for $\phi = \pi/2$ we get the inclusion

$$\{2 \cosh t \cos \theta \cosh t \exp(i\theta) : t \geq 0\} = \{\lambda \exp(i\theta) : \lambda \geq 2 \cos \theta\} \subset W_C^f(A).$$
Therefore the proposition follows from (3.2), (3.3), (3.4). \[ \square \]

**Proposition 3.2.** If \( C = -E_{12} - E_{22} \) and \( A = E_{11} + E_{12} \), then \( W_C^J(A) = \mathbb{C}\setminus(-\infty, -1] \).

**Proof.** We have

\[
W_C^J(A) = \{ (\alpha + \beta \bar{\beta}) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}
\]

Choosing \( \phi = \pi/2 \) and letting \( \theta \neq 0 \) vary in \([-\pi/2, \pi/2]\), we conclude that

\[
(3.5) \quad \{ 2 \sinh t \cos \theta \sinh t \exp(i\theta) : t \geq 0 \} = \{ \lambda \exp(i\theta) : \lambda \leq 0 \} \subset W_C^J(A),
\]

and considering \( \phi = \pi \), we find

\[
(3.6) \quad \{ 2(-\cosh t + \sinh t \cos \theta) \sinh t \exp(i\theta) : t \geq 0 \} = \{ \lambda \exp(i\theta) : \lambda \leq 0 \} \subset W_C^J(A).
\]

It follows from (3.5), (3.6) that every complex number with nonvanishing imaginary part belongs to \( W_C^J(A) \).

Taking \( \theta = \phi = 0 \) and \( t \rightarrow +\infty \), we conclude that \( [0, +\infty) \subset W_C^J(A) \), because \( 2 \sinh^2 t \in W_C^J(A) \). For a fixed \( \phi \) different from 0 or \( \pi \), we consider the real valued function of real variable \( f(t) = 2(\cosh t \cos \phi + \sinh t) \sinh t \). Its derivative \( f'(t) = (\exp(2t) + \exp(-2t)) \cos \phi + \exp(2t) - \exp(-2t) \) vanishes at \( \exp(2t) = \sqrt{(1 - \cos \phi)/(1 + \cos \phi)} \), and the function attains here the minimum value \(-1 + \sqrt{1 - \cos^2 \phi} \). Thus, the proposition follows. \[ \square \]

**Proposition 3.3.** If \( A = C = E_{11} - E_{12} \), then \( W_C^J(A) = \mathbb{C}\setminus(-\infty, 0] \).

**Proof.** We easily find

\[
W_C^J(A) = \{ (\alpha - \beta) \bar{\alpha} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}
\]

\[
= \{ (\cosh t + \sinh t \exp(i\theta))(\cosh t + \sinh t \exp(i\phi)) : 0 \leq t, \ 0 \leq \theta, \phi \leq 2\pi \}.
\]

Setting

\[
G = \{ (\cosh t + \sinh t \exp(i\theta)) : 0 \leq t, \ 0 \leq \theta \leq 2\pi \},
\]

we clearly have

\[
\{ z^2 : z \in G \} \subset W_C^J(A) = \{ z_1 z_2 : z_1, z_2 \in G \}.
\]

We show that \( G \) is the family of circles \( (x - r)^2 + y^2 = r^2 - 1, \ r \geq 1 \). In fact, if \( (x, y) \in G \), then \( 2ax = x^2 + y^2 + 1 \) and so \( x > 0 \). Conversely, if \( x > 0 \) and \( y \in \mathbb{R} \), then the real number \( r = \frac{1}{2} \left( x + \frac{1}{2} \right) + \frac{x^2}{2} \) satisfies \( (x - r)^2 + y^2 = r^2 - 1 \). Thus, \( G = \{ z \in \mathbb{C} : \Re(z) > 0 \} \). \[ \square \]
The Numerical Range of Linear Operators on the 2-Dimensional Krein Space

**Proposition 3.4.** If \( C = E_{11} - E_{12} \) and \( A = E_{21} - E_{22} \), then \( W'_C(A) = \mathbb{C}\setminus(-\infty,0) \cup \{-1\} \).

**Proof.** We obtain
\[
W'_C(A) = \{\sinh t + \cosh t \exp(i\theta) \mid t \in \mathbb{R}, 0 \leq \theta, \phi \leq 2\pi\}.
\]

By similar arguments to those used in Proposition 3.3, it can be easily seen that \( \sinh t + \cosh t \exp(i\theta), t \in \mathbb{R}, 0 \leq \theta \leq 2\pi \), ranges over the complement of the imaginary axis, taken in the complex plane, with \( i \) and \(-i\) deleted. \( \square \)

4. **The second case.** In this case, we may assume that \( \tau = (1,0)^T, \zeta = (1,1)^T \), \( \kappa = (k_1, -k_2)^T, \eta = (1,\bar{q})^T, k_1, k_2, q \in \mathbb{C}, |k_1| = |k_2| = |q| = 1 \), and we have
\[
W'_C(A) = \{(k_1\alpha + k_2\bar{\alpha} + k_1\bar{\beta} + k_2\beta)(\bar{\alpha} + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}.
\]
Writing \( k_1 = \exp(i s) \exp(i\theta) \), \( k_2 = \exp(i s) \exp(-i\theta) \), \( s, \theta \in \mathbb{R} \), it follows that
\[
W'_C(A) = \{2 \exp(i s) \Re(\exp(-i\theta)(\bar{\alpha} + \beta))(\bar{\alpha} + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}
\]
and so we may assume that \( k_1 = 1 \).

**Proposition 4.1.** Let \( C = (E_{11} + E_{21}) - \bar{\eta}(E_{21} + E_{22}), |q| = 1 \), and let \( A = E_{11} + E_{12} \). The following hold:

1) If \( q = 1 \), then \( W'_C(A) = \{0\} \cup \{z \in \mathbb{C} : \Re(z) > 0\} \).
2) If \( q \neq 1 \), then \( W'_C(A) = \mathbb{C}\setminus\{\lambda(1-q) : \lambda \in \mathbb{R}\} \cup \{1-q\} \).

**Proof.** 1) It may be easily seen that the origin belongs to \( W'_C(A) \). We have
\[
\Re(2\Re(\alpha + \beta)(\alpha + \beta)) = 2\Re(\alpha + \beta)^2 \geq 0
\]
with occurrence of equality if and only if \( \Re(\alpha + \beta) = 0 \). Thus,
\[
W'_C(A) \subset \{0\} \cup \{z \in \mathbb{C} : \Re(z) > 0\}.
\]
We prove that the reverse inclusion holds. For every \( z \in \mathbb{C} \) with \( \theta = \text{Arg}(z) \in (-\pi/2, \pi/2) \), we may find \( \alpha, \beta \in \mathbb{C} \) such that \( |\alpha|^2 - |\beta|^2 = 1 \) and \( z = 2\Re(\alpha + \beta)(\alpha + \beta) \). In fact, let \( \alpha = \cosh t \exp(i\theta) \) and \( \beta = \sinh t \exp(i\theta) \), \( t \in \mathbb{R} \), so that \( 2\exp(2t) \cos \theta = |z| \). Since \( \cos \theta > 0 \), we have \( 2\exp(2t) \exp(i\theta) \cos \theta = z \).

2) Let \( q = q_1 + iq_2 \neq 1 \). From (4.1) we easily get
\[
W'_C(A) = \{2\Re(z) w : z, w \in \mathbb{C}, 2\Re(z\bar{w}) - 2\Re(q z\bar{w}) = |1 - q|^2\},
\]
that is, the elements of $W_C^J(A)$ are the complex numbers of the form $2x_0(x + iy)$ such that

\[(4.2) \quad (q_2 x + (1 - q_1) y) y_0 = (1/2)[1 - q^2] - (1 - q_1) x_0 x + q_2 x_0 y, \quad x_0, y_0, x, y \in \mathbb{R}.
\]

If $q_2 x + (1 - q_1) y \neq 0$, then $y_0$ may always be found such that (4.2) is satisfied, and thus

$$S = \{(x, y) \in \mathbb{R}^2 : q_2 x + (1 - q_1) y \neq 0\} \subset W_C^J(A).$$

Moreover, $\Re(-i(x + iy)(1 - q)) = q_2 x + (1 - q_1) y = 0$ if and only if $z = x + iy = (1 - q), \lambda \in \mathbb{R}$. Clearly,

$$S = C\{\lambda(1 - q) : \lambda \in \mathbb{R}\}.$$

Some calculations show that there exist complex numbers $\alpha, \beta$ satisfying $\lambda(1 - q) = 2\Re(\alpha + \beta)(\alpha + q \beta)$ and $|\alpha|^2 - |\beta|^2 = 1$ if and only if $\lambda = 1$. Finally, it can easily be seen that $1 - q$ belongs to $W_C^J(A)$. \[\square\]

## 5. The sixth case.

In this case, we may assume that $\kappa$ is neutral. By replacing the inner product $[\xi_1, \xi_2]$ by $[\xi_1, \xi_2]$, we may consider (1): $\zeta = \tau = (1, 0)^T$ or (2): $\zeta = (0, 1)^T, \tau = (1, 0)^T$. In either case, we may suppose that $\kappa = (1, 1)^T$, by replacing $A$ by $\text{diag}(1, \exp(i\theta)) A \text{diag}(1, \exp(-i\theta))$ for some $\theta \in \mathbb{R}$. By performing a transformation of this type, the components of the vector $\eta = (\eta_1, \eta_2)$ may be assumed to be real and such that $|\eta_1| \neq |\eta_2|$. Thus, one of the following situations occurs:

1st Subcase. $A\xi = [\xi, (1, 1)^T](1, 0)^T, C\xi = [\xi, (1, q)^T](1, 0)^T, -1 < q < 1$.

2nd Subcase. $A\xi = [\xi, (1, 1)^T](1, 0)^T, C\xi = [\xi, (q, 1)^T](1, 0)^T, -1 < q < 1$.

3rd Subcase. $A\xi = [\xi, (1, 1)^T](1, 0)^T, C\xi = [\xi, (1, q)^T](0, 1)^T, -1 < q < 1$.

4th Subcase. $A\xi = [\xi, (1, 1)^T](1, 0)^T, C\xi = [\xi, (q, 1)^T](0, 1)^T, -1 < q < 1$.

Firstly, we treat the 1st Subcase.

**Proposition 5.1.** If $C = E_{11} - qE_{12}$ and $A = E_{11} - E_{12}$, then

$$W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : x > 1/2 - |q|/2 \cosh t, y = \sqrt{1 - q^2/2 \sinh t}, t \in \mathbb{R}\}.$$

**Proof.** By some computations we get

$$W_C^J(A) = \{(\alpha - \beta)(\pi + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} = \{(\cosh t + \sinh t \exp(i\theta))(\cosh t + q \sinh t \exp(i\phi)) : t \geq 0, 0 \leq \theta, \phi \leq 2\pi\}.$$

The case $q = 0$ is easily treated, so we may assume that $0 < q < 1$. The above set is the union of the family of circles centered at $z = z(t, \theta) = x_0 + iy_0 = \cosh^2 t +
The Numerical Range of Linear Operators on the 2-Dimensional Krein Space

\[ \sinh t \cosh t \exp(i\theta) \] whose radii \( r = r(t, \theta) \) satisfy \( R = r^2 = q^2 \sinh^2 t (1 + 2 \sinh^2 t + 2 \sinh t \cosh t \cos \theta) \), and \( R = q^2 ((x_0 - 1)^2 + y_0^2) \). It may be shown that the centers of these circles describe the open half-plane \( x > 1/2 \). Thus,

\[ W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = q^2((x_0 - 1)^2 + y_0^2), x_0 > 1/2, y_0 \in \mathbb{R} \}. \]

The boundary of \( W_C^J(A) \) is the envelope of the family of circles when their centers run over \( x = 1/2 \) (cf. Proposition 2.3 in [14]) and the result follows. □

Next, the 2nd Sub-case is studied.

**Proposition 5.2.** If \( C = qE_{11} - E_{12}, -1 < q < 1, A = E_{11} - E_{12}, \) then \( W_C^J(A) = \mathbb{C} \).

**Proof.** Some computations yield

\[ W_C^J(A) = \{(\alpha - \beta)(q\pi + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}
\]

\[ = \{(\cosh t + \sinh t \exp(i\theta))(q \cosh t + \sinh t \exp(i\phi)) : t \geq 0, \theta, \phi \leq 2\pi \}, \]

where we may assume \( q > 0 \). Therefore, \( W_C^J(A) \) is the union of the family of circles centered at \( z(t, \theta) = x_0 + i\theta y_0 = q (\cosh t + \sinh t \exp(i\theta)) \cosh t \), whose radii \( r = r(t, \theta) \) satisfy \( R = r^2 = \sinh^2 t (1 + 2 \sinh^2 t + 2 \sinh t \cosh t \cos \theta) \), and the following relation holds

\[ R = \frac{1}{q^2}((x_0 - q)^2 + y_0^2). \]

The centers of the circles describe the open half-plane \( x > q/2 \). Thus,

\[ W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = \frac{1}{q^2}(x_0 - q)^2 + y_0^2, x_0 > q/2, y_0 \in \mathbb{R} \}. \]

Clearly, we have

\[ (x - x_0)^2 + (y - y_0)^2 = \frac{1}{q^2}((x_0 - q)^2 + y_0^2). \]

If \( (x - x_0)^2 = \frac{1}{q^2}(x_0 - q)^2 \) and \( (y - y_0)^2 = \frac{1}{q^2}y_0^2 \), then \( (x, y) \) belongs to \( W_C^J(A) \). It is always possible to find a real \( y_0 \) such that the first equation holds, while the second equation is satisfied if \( x_0 = \frac{q}{1 + q^2} (1 \pm x) \). Since \( x_0 > q/2 \), this equation has a real solution \( x_0 \) for \( x > (q - 1)/2 \) and also for \( x < (1 + q)/2 \). □

The 3rd Sub-case is investigated in the following.

**Proposition 5.3.** If \( C = E_{12} - qE_{22}, -1 < q < 1, \) and \( A = E_{11} - E_{12}, \) then

\[ W_C^J(A) = \left\{ (x, y) \in \mathbb{R}^2 : (x + q/2)^2 + \frac{y^2}{(1 - q^2)} > \frac{1}{4} \right\}. \]
Proof. We easily find

\[ W_C^J(A) = \{(\bar{\beta} - \bar{\alpha})(q\bar{\alpha} + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \]

\[ = \{(\sinh t + \cosh t \exp(i\theta))(q \sinh t + \cosh t \exp(i\phi)) : t \geq 0, 0 \leq \theta, \phi \leq 2\pi\}, \]

and we may consider \( q \geq 0 \). Since \( W_C^J(A) \) is invariant under rotations around the origin in the case \( q = 0 \), we may concentrate on the case \( q > 0 \). Thus, \( W_C^J(A) \) is the union of the family of circles centered at \( z = x_0 + iy_0 = q \sinh t \) and whose radii \( r = r(t, \theta) \) satisfy \( R = r^2 = \cosh^2 t (1 + 2 \sinh^2 t + 2 \sinh t \cosh t \cos \theta) \).

The following relation holds

\[ R = \frac{1}{q^2}((x + q)^2 + y^2). \]

The centers \( z = z(t, \theta) \) of the circles describe the half-plane \( x > -q/2 \). Thus,

\[ W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = 1/q^2 ((x_0 + q)^2 + y_0^2), \]

\[ x_0 > -q/2, y_0 \in \mathbb{R}\}. \]

For a fixed complex number \( x + iy \), we consider the Apollonius circle

\[ \{(X, Y) \in \mathbb{R}^2 : (X + q)^2 + Y^2 = q^2((X - x)^2 + (Y - y)^2) \}

with center \((-q(1+qx)/1-q^2, -q^2y/1-q^2)\), and radius \( \frac{q}{1-q^2}\sqrt{(x+q)^2+y^2} \). Hence,

\[ M(x, y) = \max \{x_0 : |(x_0 + iy_0) + q(1 + qx)/1-q^2 + i^{q^2y/1-q^2} = \frac{q}{1-q^2}\sqrt{(x+q)^2+y^2} \}
\]

\[ = \frac{q(1+qx)}{1-q^2} + \frac{q\sqrt{(x+q)^2+y^2}}{1-q^2}. \]

The point \( x + iy \) belongs to \( W_C^J(A) \) if and only if \( M(x, y) > -q/2 \). Thus, we conclude

\[ W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : 4((x+q)^2+y^2) - (1+q^2+2qx)^2 > 0\}. \]

The 4th Subcase is analysed in the following.

**Proposition 5.4.** If \( C = qE_{21} - E_{22} \), \(-1 < q < 1\), and \( A = E_{11} - E_{12} \), then \( W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : y = \sqrt{1-q^2/2} \sinh t, x > -1/2 - |q|/2 \cosh t, t \in \mathbb{R}\}. \)

**Proof.** We find

\[ W_C^J(A) = \{(\bar{\beta} - \bar{\alpha})(q\bar{\alpha} + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \]

\[ = \{(\sinh t + \cosh t \exp(i\theta))(\sinh t + q \cosh t \exp(i\phi)) : t \geq 0, 0 \leq \theta, \phi \leq 2\pi\}. \]
The case \( q = 0 \) is trivial, so we assume \( 0 < q < 1 \). Under this assumption, \( W_C^I(A) \) is the union of the family of circles centered at \( z = z(t, \theta) = x + iy = \sinh t \sinh(t + \cosh t \exp(i \theta)) \), whose radii \( r = r(t, \theta) \) satisfy \( R = r^2 = q^2 \cosh^2(1 + 2 \sinh^2 t + 2 \sinh t \cosh t \cos \theta) \), and also \( R = q^2 ((x + 1)^2 + y^2) \). The centers of the circles range over the half-plane \( x > -1/2 \) and the proposition easily follows from Proposition 5.1. □

6. The third case. In this case, we are assuming that \( \kappa, \tau \) are neutral and \( \eta, \zeta \) are non-neutral. We may consider \( \zeta = (1, 0)^T, \tau = (1, -1)^T, \kappa = (1, -k), k \in \mathbb{C} \) with \( |k| = 1 \) and \( \eta = (\eta_1, \eta_2), \eta_1, \eta_2 \in \mathbb{R}, |\eta_1| \neq |\eta_2| \). Under these assumptions, we obtain

\[
W_C^I(A) = \{ (\pi + k\beta)(\eta_1 \alpha + \beta) + \eta_2(\pi + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}.
\]

Firstly, we consider the three special Subcases: (1) \( \eta_2 = 0, \eta_1 = 1; \) (2) \( \eta_1 = 0, \eta_2 = 1; \) (3) \( k = 1 \). Finally, in Proposition 6.4 we treat the case \( k \neq 1 \) and \( \eta_1 \neq \eta_2 \neq 0 \).

If the Subcase (1) occurs, then

\[
W_C^I(A) = \{ (\alpha + \beta)(\alpha + k\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}.
\]

If \( k = 1 \), then \( W_C^I(A) \) is the positive real line. Let \( k \neq 1 \) and let \( k_1 \) be a complex number such that \( k_1^2 = k \). Thus, \( |k_1| = 1, k_1 \neq 1, k_1 \neq -1 \) and

\[
W_C^I(A) = \overline{k_1}\{ (\alpha + \beta)(k_1 \alpha + k_1 \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}
= \overline{k_1}\{ k_1 + (k_1 + \overline{k_1})|\beta|^2 t + 2\Re(k_1 \overline{\alpha}\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}.
\]

Multiplying \( W_C^I(A) \) by a complex number \( k_1 \), and performing some calculations, the next proposition follows.

**Proposition 6.1.** Let \( C = E_{11}, A = k_1(E_{11} - E_{21}) + \overline{k_1}(E_{21} - E_{22}), k_1 = \exp(i\phi), -\pi < \phi < \pi, \phi \neq 0 \). Then \( W_C^I(A) = \{ i \sin \phi + t : t \in \mathbb{R} \} \).

In the Subcase (2), we replace \((1, -k)^T\) by \((1, -\overline{k})^T\) and we prove.

**Proposition 6.2.** Let \( C = -E_{12}, A = (E_{11} - E_{21}) + k(E_{12} - E_{22}), k = \exp(i\phi), -\pi < \phi \leq \pi \). Then \( W_C^I(A) = \{ z \in \mathbb{C} : |z| \geq \sin(|\phi|/2) \} \) for \( \phi \neq 0 \), and \( W_C^I(A) = \mathbb{C}\setminus\{0\} \) for \( \phi = 0 \).

**Proof.** We get

\[
W_C^I(A) = \{ (\alpha + k\beta)(\alpha + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \},
\]

being this set invariant under the rotation \( z \mapsto \exp(i\theta)z, \theta \in \mathbb{R} \). Therefore, to conclude the proof it is sufficient to show that the function defined on \([0, +\infty) \times [0, 2\pi]\) by \( f(t, \theta) = |\cosh t + \exp(i\phi) \exp(i\theta) \sinh t|^2 |\cosh t + \exp(i\theta) \sinh t|^2 \) ranges over
Next we consider the Subcase (3).

**Proposition 6.3.** Let \( C = E_{11} - qE_{12}, \ q \in \mathbb{R}, \ q \neq 1, \ q \neq -1, \) and let \( A = E_{11} + E_{12} - E_{21} - E_{22}. \) The following hold:

1) If \( |q| > 1, \) then \( W^L_C(A) = \mathbb{C}\setminus\{0\}. \)

2) If \(-1 < q < 1, \) then \( W^L_C(A) = \{ (x, y) \in \mathbb{R}^2 : x > 0, \ |y| \leq |q| x / \sqrt{1 - q^2} \}. \)

**Proof.** Performing some computations, we obtain

\[
W^L_C(A) = \{ (\alpha + \beta)(\pi + k + q \alpha + q \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}
\]

= \{ |z| + |q| z : z \in \mathbb{C}\setminus\{0\} \}. \]

1) Let \( |q| > 1. \) The equation \( |z| + |q| z = 0 \) holds if and only if \( z \neq 0. \) Given a non-zero complex number \( z_0 = x_0 + iy_0 \) it is always possible to find a complex \( z \) satisfying \( z_0 = |z| + |q| z, \) since the real system \( x_0 = \sqrt{x^2 + y^2 + |q| x}, \ y_0 = |q| y \) is possible.

2) Let \(-1 < q < 1. \) We may assume \( q \neq 0. \) It can be easily checked that the equations of the above system are satisfied for real numbers \( x_0, y_0 \) if and only if \( x_0 > 0 \) and \( \frac{y_0}{x_0} < \frac{q^2}{1 - q^2}. \) \]

**Proposition 6.4.** Let \( C = E_{11} - qE_{12}, \ q \in \mathbb{R}, \ q \neq 1, \ q \neq -1, \) \( A = k_1(E_{11} - E_{21}) + k_1(E_{12} - E_{22}), \ k_1 = \exp(i \psi), \ -\pi < \psi < \pi, \) then \( \phi, \theta \neq 0. \) The following hold:

1) If \( |q| > 1, \) then

\[
W^L_C(A) = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{q^2 - 1} + \frac{(y - \sin \psi)^2}{q^2} \geq \sin^2(\psi) \right\}.
\]

2) If \( 0 < |q| < 1, \) then

\[
W^L_C(A) = \left\{ (x, y) \in \mathbb{R}^2 : \frac{(y - \sin \psi)^2}{q^2} - \frac{x^2}{1 - q^2} \leq \sin^2(\psi) \right\}.
\]

**Proof.** Some computations yield

\[
W^L_C(A) = \{ (k_1 \alpha + k_1 \beta)(\pi + k + qa + q \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}
\]

= \{ [(k_1 \cosh t + \sqrt{k_1} \sinh t \exp(i \phi)][\cosh t + \sinh t \exp(-i \phi)] + q [k_1 \cosh t + \sqrt{k_1} \sinh t \exp(i \phi)][\cosh t + \sinh t \exp(i \phi)] \exp(i \theta) : t \geq 0, 0 \leq \phi, \theta \leq 2\pi \}.\]
Then, \( W_C'(A) \) is the union of the family of circles centered at

\[
z = z(t, \phi, \psi) = x + i y = [(k_1 \cosh t + k_1 \sinh t \exp(i\phi))] [\cosh t + \sinh t \exp(-i\phi)]
\]

\[
= k_1 + 2 \cos(\psi) \sinh^2 t + 2 \sinh t \cos \phi - \psi,
\]

where \( \psi = \arg k_1 \). The radii of the above family, \( r = r(t, \theta) \), satisfy \( R = r^2 = q^2 \cosh^2 t + \sinh^2 t + 2 \sinh t \cos \phi - 2 \psi \). Thus, \( R = q^2 (x^2 + \sin^2(\psi)) \), \( y = \sin \psi \), and we easily get

\[
W_C'(A) = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - \sin \psi)^2 = q^2 (x_0^2 + \sin^2 \psi), x \in \mathbb{R}\}.
\]

1) Let \(|q| > 1\). If \( x + iy \) is a boundary point of \( W_C'(A) \), then it satisfies

\[
F(x, y; x_0) = (x - x_0)^2 + (y - \sin \psi)^2 - q^2 (x_0^2 + \sin^2 \psi) = 0,
\]

and \( F_x(x, y; x_0) = -2 (x + (q^2 - 1)x_0) = 0 \). By eliminating \( x_0 \) in the above equations, we get

\[
\frac{x^2}{q^2 - 1} + \frac{(y - \sin \psi)^2}{q^2} = \sin^2 \psi,
\]

and so the boundary of \( W_C'(A) \) is contained in this ellipse. Conversely, we claim that every point of this ellipse is contained in the boundary of \( W_C'(A) \). Indeed, every point \((x, y)\) of the ellipse is parametrically represented as \( x = \sqrt{(q^2 - 1) \sin \psi \cos \theta}, y = \sin \psi + q \sin \psi \sin \theta, \theta \in \mathbb{R} \), and satisfies

\[
\left( x + \frac{\sin \psi \cos \theta}{\sqrt{q^2 - 1}} \right)^2 + (y - \sin \psi)^2 - q^2 \left( \frac{\sin^2 \psi \cos^2 \theta}{q^2 - 1} + \sin^2 \psi \right) = 0.
\]

Thus, \((x, y)\) is an element of (6.2). To finish the proof, we observe that if \( x_0 \to \infty \), then the circle \( F(x, y; x_0) = 0 \) contains points \( x + i0 \) with \( x \to \infty \). Let \( 0 < |q| < 1 \). By similar arguments to the above, we find that the boundary of \( W_C'(A) \) is the hyperbola:

\[
\frac{(y - \sin \psi)^2}{q^2} - \frac{x^2}{1 - q^2} = \sin^2(\psi).
\]

If \( x_0 \to +\infty \), then the circle \( F(x, y; x_0) = 0 \) contains points \( x \) of the real line with \( x \to +\infty \).

7. Proof of main theorems and a concluding remark. As a consequence of the descriptions of the shape of \( W_C'(A) \) in Sections 2–6 and [14], we obtain Theorem 1.1. We remark that \( W_C'(A) \), or each connected component of its complement \( \mathbb{C} \setminus W_C'(A) \), is convex. This property does not hold for 3-dimensional Krein spaces. From Theorem 1.2 it follows that \( W_C'(A) \) has at most 1 hole.
REFERENCES