2012

The characteristic set with respect to the k-maximal vectors of a tree

Shi-Cai Gong
scgong@zafu.edu.cn

Follow this and additional works at: http://repository.uwyo.edu/ela

Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1505

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in Electronic Journal of Linear Algebra by an authorized editor of Wyoming Scholars Repository. For more information, please contact scholcom@uwyo.edu.
THE CHARACTERISTIC SET WITH RESPECT TO 
K-MAXIMAL VECTORS OF A TREE*

SHI-CAI GONG†

Abstract. Let T be a tree on n vertices and \( L(T) \) be its Laplacian matrix. The eigenvalues and eigenvectors of T are respectively referred to those of \( L(T) \). With respect to a given eigenvector \( Y \) of T, a vertex \( u \) of T is called a characteristic vertex if \( Y[u] = 0 \) and there is a vertex \( w \) adjacent to \( u \) with \( Y[w] \neq 0 \); an edge \( e = (u, w) \) of T is called a characteristic edge if \( Y[u]Y[w] < 0 \). \( C(T, Y) \) denotes the characteristic set of T with respect to the vector Y, which is defined as the collection of all characteristic vertices and characteristic edges of T corresponding to Y.

Let \( \lambda_1(T) \leq \lambda_2(T) \leq \cdots \leq \lambda_n(T) \) be the eigenvalues of a tree T on n vertices. An eigenvector is called a \( k \)-vector (\( k \geq 2 \)) of T if the eigenvalue \( \lambda_k(T) \) associated by this eigenvector satisfies \( \lambda_k(T) > \lambda_{k-1}(T) \). The \( k \)-vector \( Y \) of T is called \( k \)-maximal if \( C(T, Y) \) has maximum cardinality among all \( k \)-vectors of T. In this paper, the characteristic set with respect to any \( k \)-maximal vector of a tree is investigated by exploiting the relationship between the cardinality of the characteristic set and the structure of this tree. With respect to any \( k \)-maximal vector \( Y \) of a tree T, the structure of the trees T satisfying \( |C(T, Y)| = k - 1 - t \) for any \( t (0 \leq t \leq k - 2) \) are characterized.

Key words. Laplacian matrix, Characteristic set, \( k \)-Vector, \( k \)-Maximal vector.

AMS subject classifications. 05C50, 15A15.

1. Introduction. Let \( G = (V, E) \) be a simple graph with vertex set \( V = V(G) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E = E(G) \). The Laplacian matrix of \( G \) is defined as \( L = L(G) = D(G) - A(G) \), where \( A(G) \) is the adjacency matrix of \( G \) and \( D(G) = \text{diag}(d(v_1), d(v_2), \ldots, d(v_n)) \), the diagonal degree matrix of \( G \). Since \( L(G) \) is positive semi-definite, its eigenvalues can be arranged as

\[
0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G).
\]

Henceforth \( \lambda_i(G) \) denotes the i\(th\) smallest eigenvalue of \( G \). The \( k \)th smallest eigenvalue of \( G \) will be written as \( k\lambda(G) \) if \( \lambda_k(G) > \lambda_{k-1}(G) \), and the corresponding eigenvectors will be called \( k \)-vectors of \( G \).

For an eigenvector \( Y \) of a given graph \( G \), a vertex \( v \) is called a characteristic vertex with respect to \( Y \) if \( Y[v] = 0 \) and there is a vertex \( w \) adjacent to \( v \), such that \( Y[w] \neq 0 \);

---

*Received by the editors on September 9, 2010. Accepted for publication on December 18, 2011. Handling Editor: Bryan L. Shader.
†School of Science, Zhejiang A & F University, Hangzhou, 311300, China (scgong@zafu.edu.cn). Supported by NSF of China (11171373, 10871230), NSF of Department of Education of Anhui (KJ2010A092), and NSF of Zhejiang (Y7080364, Y607480).
an edge $e = (u, w)$ is called a characteristic edge of $G$ with respect to $Y$ if $Y[u] Y[w] < 0$. We denote by $C(G, Y)$ the characteristic set of $G$ with respect to the vector $Y$, which is defined as the collection of all characteristic vertices and characteristic edges of $G$ corresponding to $Y$. For convenience we relax the requirement that $Y$ be an eigenvector of $G$ in the definition of $C(G, Y)$, and allow $Y$ to be an arbitrary vector defined on the vertex set of $G$.

For a graph $G$, an eigenvector corresponding to the second smallest eigenvalue is called a Fiedler vector of $G$. It is known that $\lambda_2(G) > \lambda_1(G) = 0$ if and only if $G$ is connected [5]. Thus, each Fiedler vector of a connected graph is a $2$-vector. Fiedler's remarkable result [5, Theorem 3.14] on the structure of Fiedler vectors (i.e., $2$-vectors) of a connected graph motivated a lot of work on the structure of eigenvectors; see, e.g., [1, 2, 7, 8, 9, 10, 11, 12, 13, 14].

Merris introduced the notion of a characteristic set and showed that $|C(T, Y)| = 1$. In [11], Merris also showed that $C(T, Y)$ is fixed regardless of the choice of Fiedler vectors $Y$ of a given tree $T$; see [11, Theorem 2]. With respect to any Fiedler vectors $Y$ of a given graph $G$, Bapat and Pati [1] investigated the cardinality of the characteristic set $C(G, Y)$. In [14], Pati extended the notation the characteristic set from Fiedler vectors to $3$-vectors of trees and gave a complete description of $3$-vectors of a given tree. Then Fan and Gong [2] further extended the concept of characteristic set to any $k$-vector of a tree.

Recall that, for any $2$-vector $Y$ of a tree $T$, $|C(T, Y)| = 1$ and $C(T, Y)$ is fixed regardless of the choice of $2$-vectors $Y$, even though the eigenspace for $2\lambda(T)$ (well known as the algebraic connectivity of $T$) is large (see [11], Theorem 2).

However, for $k \geq 3$, the characteristic set $C(T, Y)$ may depend upon the choice of the $k$-vectors. For example, consider the tree $T$ in Figure 1.1 (or see Figure 3.2 in [14]). One can find that $Y_1, Y_2$ and $Y_3$ are all $3$-vectors of $T$, where $Y_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2638 & 0.4754 & 0.5929 & 0 & 0 & -0.2638 & -0.4754 & -0.5929 \end{bmatrix}^T$, $Y_2 = \begin{bmatrix} 0.5929 & 0.4754 & 0.2638 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5929 & -0.4754 & -0.2638 & 0 & 0 & 0 \end{bmatrix}^T$, and $Y_3 = \begin{bmatrix} 0.9098 & 0.7296 & 0.4049 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.9098 & -0.7296 & -0.4049 & 0 & 0 & 0 \end{bmatrix}^T$.

But one can verify that $C(T, Y_1) = \{11\}, C(T, Y_2) = \{4\}$, and $|C(T, Y_3)| = \{4, 11\}$. For a given tree $T$ and a $k$-vector $\overline{\mathbf{Y}}$ of $T$, $\overline{\mathbf{Y}}$ is called $k$-maximal if $C(T, \overline{\mathbf{Y}})$ has maximum cardinality over all $k$-vectors of $T$, i.e.,

$$|C(T, \overline{\mathbf{Y}})| = \max_{\overline{\mathbf{Y}}} |C(T, \overline{\mathbf{Y}})|,$$

where the maximum is taken over all $k$-vectors of $T$ (see [6]).
For any $k$-maximal vector of a tree, the following result is interesting.

**Proposition 1.1.** ([6], Theorem 3.2) Let $T$ be a tree on $n$ vertices. Suppose that both $Y_1$ and $Y_2$ are the $k(\geq 2)$-maximal vectors of $T$. Then

$$C(T, Y_1) = C(T, Y_2).$$

Proposition 1.1 implies that for any $k$ with $2 \leq k \leq n$, the characteristic set $C(T, Y)$ is fixed regardless the choice of the $k$-maximal vector $Y$, i.e., the characteristic set is determined by the tree structure and independent of the $k$-maximal vectors, which is consistent with Merris’ result (see [11], Theorem 2). Henceforth, to exploit the relationship between the cardinality of the characteristic set and the tree structure, we focus on studying the $k$-maximal vectors of trees.

With respect to any $k$-(maximal) vector $Y$ of a given tree $T$, Fan et al. showed that [2, Corollary 2.5]

$$1 \leq |C(T, Y)| \leq k - 1. \quad (1.1)$$

In particular, they also gave a characterization for trees whose characteristic set $C(T, Y)$ with respect to its any $k$-vector $Y$ contains exactly one element, i.e., the $k$-simple trees; see [2, Theorem 2.11]. Naturally, the following problem is posed:

For a general tree $T$ on $n$ vertices and an arbitrary integer $k (\leq n)$, can we exploit the relationship between the cardinality of the characteristic set $C(T, Y)$ with respect to its any $k$-maximal vector $Y$ and the structure of such a tree $T$?

In this paper, we investigate the characteristic set with respect to any $k$-maximal vector of a given tree and consider the problem above. The rest paper is organized as follows. In Section 2, we first list several preliminary results. Then, for any $k$-(maximal) vector $Y$ of a given tree $T$, we establish some lemmas that relate characteristic vertex and the structure for the subvector of $Y$. In Section 3, we study the cardinality of the characteristic set $C(T, Y)$ with respect to any $k$-maximal vector $Y$. 
of a tree $T$, and determine the structure of the trees $T$ satisfying $|C(T, Y)| = k - 1 - t$, where $0 \leq t \leq k - 2$. In addition, examples that illustrate the occurrence of each of the case described in our theorems are given.

2. Preliminary results. Let $G$ be a connected graph on $n$ vertices, $L$, its Laplacian matrix, and $Y$, a vector defined on the vertex set of $G$. We will use following notation. For $U \subseteq V(G)$, $W \subseteq V(G)$, denote by $L[U, W]$ the submatrix of $L$ with rows corresponding to the vertices of $U$ and columns corresponding to the vertices of $W$, if $U = W$, $L[U, W]$ is simply written as $L[U]$; and similarly, denote by $Y[U]$ the subvector of $Y$ corresponding to the vertices of $U$. For convenience, we usually write $L[G_1, G_2]$ and $Y[G_1]$ instead of $L[V(G_1), V(G_2)]$ and $Y[V(G_1)]$ for subgraphs $G_1, G_2$ of $G$, respectively.

With respect to a vector $Y$ which gives a valuation of vertices of $G$, a vertex $v$ is called a zero (nonzero) vertex if $Y[v] = 0$ ($Y[v] \neq 0$), a component containing a nonzero vertex is called a nonzero component. Denote by $S(G)$ and $m_G(\lambda)$ the spectrum and the multiplicity of the eigenvalue $\lambda$ of the Laplacian matrix of a graph $G$, respectively.

Let $L$ be the Laplacian matrix of a graph $G = (V, E)$ and $Y$, an eigenvector of $L$ corresponding to the eigenvalue $\lambda$. Then the eigencondition at the vertex $v$ is the equation

$$\sum_{(i, v) \in E} L[i, v]Y[i] = (\lambda - L[v, v])Y[v].$$

An $n \times n$ matrix $A$ will be called acyclic if it is symmetric and if for any mutually distinct indices $k_1, k_2, \ldots, k_s$ ($s \geq 3$) in $\{1, 2, \ldots, n\}$, the equality

$$A[k_1, k_2]A[k_2, k_3] \cdots A[k_s, k_1] = 0$$

is fulfilled. Then the Laplacian matrix of a tree is acyclic. Denote by $m_A^+(\lambda)$ (respectively, $m_A^-(-\lambda)$) the number of eigenvalues of the matrix $A$ greater than (respectively, less than) $\lambda$, and let $m_A(\lambda)$ the multiplicity of $\lambda$. The following results are known from the work of Fiedler.

**Lemma 2.1.** ([4], Lemma 1.12) Let

$$A = \begin{bmatrix} B & C \\ C^T & d \end{bmatrix}$$

be a partitioned symmetric real matrix, where $C$ is a vector. If there exists a vector $U$ such that $BU = 0$ and $C^TU \neq 0$. Then

$$m_A^-(0) = m_B^-(0) + 1 \quad \text{and} \quad m_A^+(0) = m_B^+(0) + 1.$$
Lemma 2.2. ([4], Theorem 2.3) Let $A$ be an $n \times n$ acyclic matrix. Let $Y$ be an eigenvector of $A$ corresponding to an eigenvalue $\lambda$.

Let there first be no “isolated” zero coordinate of $Y$, that is coordinate $Y[k] = 0$ such that $A[k,j]Y[j] = 0$ for all $j$. Then

$$m^+_{A}(\lambda) = a^+ + r, \quad m^-_{A}(\lambda) = a^- + r,$$

where $r$ is the number of zero coordinates of $Y$, $a^+$ is the number of those unordered pairs $(i,k)$ for which $A[i,k]Y[i]Y[k] < 0$ and $a^-$ is the number of those unordered pairs $(i,k)$ ($i \neq k$), for which $A[i,k]Y[i]Y[k] > 0$.

If there are isolated zero coordinates of $Y$, $M$ is the set of indices corresponding to such coordinates and $A'$ the matrix obtained from $A$ by deleting all rows and columns with indices from $M$, then the numbers $m^+_{A}(\lambda)$, $m^-_{A}(\lambda)$ and $m_A(\lambda)$ satisfy

$$m^+_{A}(\lambda) = m^+_{A'}(\lambda) + c_1, \quad m^-_{A}(\lambda) = m^-_{A'}(\lambda) + c_2, \quad m_A(\lambda) = m_A(\lambda) + c_0,$$

where $c_1$, $c_2$ and $c_0$ are nonnegative integers such that $c_1 + c_2 + c_0 = |M|$, the number of elements in $M$.

Lemma 2.3. ([4], Corollary 2.5) Let $A$ be an $n \times n$ irreducible acyclic matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If $\lambda_r$ corresponding to an eigenvector $Y$ with all coordinates different from zero, then $\lambda_r$ is simple and there are exactly $r - 1$ (unordered) pairs $(i,k), i \neq k$, for which $A[i,k]Y[i]Y[k] > 0$.

Denote by $C_V(T,Y)$ the collection of all characteristic vertices in $C(T,Y)$ (or briefly $C_V$). From Lemma 2.3 for any $k$-vector $Y$ of a given tree $T$ on $n$ vertices, either $|C(T,Y)| = k - 1$ or $C_V(T,Y)$ contains characteristic vertices. Thus, as a consequence of Lemma 2.3 and (1.1), we have:

Corollary 2.4. Let $T$ be a tree on $n$ vertices and $Y$, a $k$-vector of $T$ with $2 \leq k \leq n$. If $|C(T,Y)| \leq k - 2$, then

$$|C_V(T,Y)| \geq 1.$$
In addition, the following two lemmas are needed for our discussion.

**Lemma 2.5.** ([6], Lemma 2.2) Let $T$ be a tree on $n$ vertices with Laplacian matrix $L$. Suppose that $\lambda \in S(T)$ and $v \in V(T)$. Let also $Y$ be an eigenvector of $L$ corresponding to $\lambda$. If $v \in C(T,Y)$, then $m_{L[T-v]}(\lambda) = m_L(\lambda) + 1$.

**Lemma 2.6.** ([6], Lemma 2.3) Let $T$ be a tree on $n$ vertices with Laplacian matrix $L$. Let also $\lambda \in S(T)$ and $v \in V(T)$. If $m_{L[T-v]}(\lambda) = m_L(\lambda) + 1$, then $Y[v] = 0$ for any eigenvector $Y$ of $L$ corresponding to $\lambda$.

Let $T = (V,E)$ be a tree on $n$ vertices with Laplacian matrix $L$ and let $\lambda$ be an nonzero eigenvalue of $L$. Suppose $W$ is a subset of $V$ and $T - W$ denotes the graph obtained from $T$ by deleting the vertices $W$ together with all edges incident to them. Suppose also that $M = \{ T_i : i = 1, 2, \ldots, m \}$ is the collection of all components of $T - W$. According to whether or not the eigenvalue $\lambda$ is contained in $S(L[T_i])$, we partition $M$ as follows:

(a) $M_1(W; \lambda) = \{ T_i : \lambda < \lambda_1(L[T_i]); T_i \in M \}$,
(b) $M_2(W; \lambda) = \{ T_i : \lambda = \lambda_1(L[T_i]); T_i \in M \}$,
(c) $M_3(W; \lambda) = \{ T_i : \lambda > \lambda_1(L[T_i]) \text{ and } \lambda \in S(L[T_i]); T_i \in M \}$, and
(d) $M_4(W; \lambda) = \{ T_i : \lambda > \lambda_1(L[T_i]) \text{ and } \lambda \notin S(L[T_i]); T_i \in M \}$.

Let $Y$ be a $k$-vector of a tree $T$, $v \in C(T,Y)$ and $T'$ a component of $T - v$. According to whether the component $T'$ is of $M_1(v;^k\lambda)$, $M_2(v;^k\lambda)$, $M_3(v;^k\lambda)$, or $M_4(v;^k\lambda)$, we establish the following structural property for $Y[T']$.

**Lemma 2.7.** Let $T$ be a tree with Laplacian matrix $L$ and let $Y$ be a $k$-vector of $T$. Suppose $v \in C(T,Y)$ and $T'$ is a component of $T$ at $v$. Then
(a) $Y[T'] = 0$ if $T' \in M_1(v;^k\lambda)$ or $T' \in M_3(v;^k\lambda)$,
(b) $Y[T']$ is either zero, or positive, or negative if $T' \in M_2(v;^k\lambda)$, and
(c) $Y[T']$ is either zero or non-zero containing both positive entries and negative entries if $T' \in M_4(v;^k\lambda)$.

**Proof.** From Lemma 2.5 we have $m_{L[T-v]}(^k\lambda) = m_L(^k\lambda) + 1$ as $v \in C(T,Y)$. Thus, $Y[v] = 0$ by Lemma 2.6. Combining with the equation $(L - ^k\lambda I)Y = 0$, we have

$$(L[T'] - ^k\lambda I)Y[T'] = 0.$$ 

Then part (a) holds, since in that case $\det(L[T'] - ^k\lambda I) \neq 0$. Note that $L[T']$ is an $M$-matrix, then the eigenvector corresponding to its least eigenvalue is either positive or negative (see, for instance, [1], Lemma 1). Consequently, part (b) follows. Part
(c) follows from the Perron-Frobenius theorem and the fact that the eigenvectors corresponding to the least eigenvalue are orthogonal to the eigenvectors corresponding to each other eigenvalue. □

Furthermore, if the k-vector Y is restricted to k-maximal, then Lemma 2.7(c) can be strengthened as follows:

**Lemma 2.8.** Let T be a tree with Laplacian matrix L and let Y be a k-maximal vector of T. Suppose that v ∈ C(T, Y) and T' is a component of T at v. If T' ∈ M3(v; k λ), then Y[T'] has at least one positive and at least one negative entry.

**Proof.** Assume to the contrary that Y[T'] = 0 by Lemma 2.7. By the definition of the k-maximal vector, it is sufficient to construct a k-vector W of L such that |C(T, W)| ≥ |C(T, Y)| + 1.

Firstly, we have

\[(2.1) \quad C(T, Y) = C(T - T', Y[T - T']) + C(T', Y[T']) = C(T - T', Y[T - T']),\]

the last equation holds from C(T', Y[T']) = 0 as Y[T'] = 0 by assumption. Note that if \(k \lambda\) is an eigenvalue of the principle submatrix L[T'], then there exists a nonzero vector, say Y', such that L[T']Y' = k \(\lambda\)Y'. Since T' ∈ M3(v; k λ), k \(\lambda\) is not the least eigenvalue of L[T']. Applying the Perron-Frobenius theorem again, we see that the vector Y' contains both positive entries and negative entries. Therefore, |C(T', Y')| ≥ 1.

Let v' ∈ T' adjacent to v. Since v ∈ C(T, Y), there exists a vertex, say v1, adjacent to v such that Y[v1] ≠ 0. Without loss of generality, suppose Y[v1] ≠ Y'[v']. (otherwise, we can replace Y' by αY' for some nonzero scalar α (≠ 1).) Let T1 be the component of T at v containing v1 and W be the vector obtained from Y by replacing Y[T'] and Y[T1] by Y' and tY[T1], respectively, in which t = (Y[v1] - Y'[v'])/Y[v1]. We can readily verify that, corresponding to the vector W, the vertex v satisfies the eigencondition. Then W is also a k-vector of T and |C(T1, Y[T1])] = |C(T1, tY[T1])] as t ≠ 0. Henceforth, with respect to the k-vector W, we have

\[
|C(T, W)| = |C(T - T', W[T - T'])| + |C(T', Y[T'])| \\
\geq |C(T - T', W[T - T'])| + 1 \\
= |C(T - T', Y[T - T'])| + 1 \\
= |C(T, Y)| + 1.
\]

The third equality follows from (2.1). □

3. The cardinality of the characteristic set with respect to k-maximal vectors of a tree. In this section, we investigate |C(T, Y)| for a k-maximal vector Y and characterize the structure of trees T with |C(T, Y)| = k - 1 - t for some t with 0 ≤ t ≤ k - 2.
Denote by $A \oplus B$ the direct sum of matrices $A$ and $B$. We begin our discussion with the result which reveals the secret why the upper bound in (1.1) for the cardinality of the characteristic set $|C(T, Y)|$ with respect to $k$-maximal vector $Y$ of a given tree $T$ is sometimes not sharp.

**Lemma 3.1.** Let $T$ be a tree with Laplacian matrix $L$ and let $Y$ be a $k$-maximal vector of $T$. Suppose that $v \in C(T, Y)$ and $M_4(v; \lambda) = \{T_i : i = 1, 2, \ldots, p; p \geq 1\}$.

\[ \mathcal{C}(T, Y) = |C(T, Y)| \leq k - 1 - t. \]

**Proof.** For convenience, let $T' = \cup_{i=1}^p T_i$. Firstly, we have

\begin{align*}
\mathcal{C}(T, Y) &= \mathcal{C}(T, Y[T', Y[T']]) + \mathcal{C}(T', Y[T']) \\
&= \mathcal{C}(T', Y[T']) - \mathcal{C}(T, \lambda[T - v]),
\end{align*}

as $Y[v] = 0$ by Lemma 2.6 and $Y[T'] = 0$ by Lemma 2.7. From Lemma 2.5, we have $m_{L[T']} k\lambda = m_{L[T]} k\lambda + 1$. Therefore,

\[ m_{L[T]} = m_{L[T']} - 1, \]

which implies that

\[ \lambda_{k-2} (L[T - v]) < \lambda_{k-1} (L[T - v]) = k \lambda. \]

Note that $S(L[T - v]) = S(L[T - T' - v]) \cup S(L[T'])$. Combining this with $t = \bigoplus_{i=1}^p m_{L[T_i]} k\lambda$, we have

\[ \lambda_{k-2} (L[T - T' - v]) < \lambda_{k-1} (L[T - T' - v]) = \lambda_{k-1} (L[T - T' - v]). \]

Applying Lemma 2.1 to the matrix $L[T - T']$, its principal submatrix $L[T - T' - v]$ and the vector $L[T - T', v]$, we have

\[ \lambda_{k-1} (L[T - T']) < k \lambda = \lambda_{k-t} (L[T - T']). \]

i.e.,

\[ m_{L[T - T']} k\lambda = k - 1 - t. \]

Furthermore, applying Lemma 2.2 to $L[T - T']$ and $Y[T - T']$, we have $m_{L[T]} k\lambda = a^+ + r$. One can find that $a^-$ and $r$ are exactly the number of characteristic edges and the characteristic vertices in $C(T - T', Y[T - T'])$, respectively. Then

\[ |C(T - T', Y[T - T'])| = m_{L[T]} k\lambda = a^+ + r \leq k - 1 - t, \]
where the last inequality follows from the fact that $L[T' - T]$ has exactly

$$k - 1 - t = k - 1 - \sum_{i=1}^{p} m_{L[T_i]}(k\lambda)$$

eigenvalues less than $k\lambda$. Thus, $|C(T, Y)| \leq k - 1 - t$ by (3.1), and the result follows.

Applying the method above repeatedly to every element of $C_{V}(T, Y)$, the following result can be obtained immediately.

**Theorem 3.2.** Let $T$ be a tree with its Laplacian matrix $L$ and let $Y$ be a $k$-maximal vector of $T$. Suppose that $M_3(C_{V}; k\lambda) = \{T_i : i = 1, 2, \ldots, p; p \geq 1\}$. Let $t = \bigoplus_{i=1}^{p} m_{L[T_i]}(k\lambda)$. Then

$$|C(T, Y)| \leq k - 1 - t.$$ 

Let $T$ be a tree on $n$ vertices with Laplacian matrix $L$, and let $Y$ be a $k$-vector of $T$. Suppose that $v \in C(T, Y)$ and $T' \in M_3(v; k\lambda)$. From Lemma 2.8, the maximum of the vector $Y$ ensures that the subvector $Y[T']$ is nonzero. In fact, as we will see in the lemma below that such a maximum even preserves the cardinality of the characteristic set $C(T', Y[T'])$ with respect to any $k$-maximal vector $Y[T']$ of $T'$; this gives the sharp upper bound in (1.1).

**Lemma 3.3.** Let $T$ be a tree with Laplacian matrix $L$ and let $Y$ be a $k$-maximal vector of $T$. Suppose that $C_{V}(T, Y) \neq \emptyset$ and $T' \in M_3(C_{V}; k\lambda)$. Then

$$|C(T', Y[T'])| = m_{L[T']}^{-}(k\lambda).$$

**Proof.** From Lemma 2.3, it is sufficient to show that all coordinates of the subvector $Y[T']$ are nonzero. Otherwise, assume that $Y[T']$ contains zero entries, then there exists a zero vertex, say $v$, adjacent to some nonzero vertex. Then, with respect to the $k$-vector $Y$, such zero vertex $v$ forms a characteristic vertex, which is a contradiction to the hypothesis that $v$ lies in the component $T'$ belonging to $T - C_{V}(T, Y)$. \(\blacksquare\)

Next we show that the upper bound in Theorem 3.2 is indeed the cardinality of the characteristic set with respect to any $k$-maximal vector described as above.

**Theorem 3.4.** Let $T$ be a tree with Laplacian matrix $L$ and let $Y$ be an arbitrary $k$-maximal vector of $T$. Suppose that $C_{V}(T, Y) \neq \emptyset$ and $M_4(C_{V}; k\lambda) = \{T_i : i = 1, 2, \ldots, p; p \geq 0\}$. Let $t = \bigoplus_{i=1}^{p} m_{L[T_i]}(k\lambda)$. Then

$$|C(T, Y)| = k - 1 - t.$$
Proof. Let $C_Y(T, Y) = \{v_1, v_2, \ldots, v_m\}$ and $\{T_i : i = 1, 2, \ldots, p, p + 1, p + 2, \ldots, p + l\}$ be all components of $T - C_Y$. Obviously $1 \leq m \leq |C(T, Y)|$. Note that $T$ contains no cycles. Thus each vertex $v_i \in C_Y(T, Y)$ is adjacent to at least two nonzero components, and each pair of characteristic vertices is adjacent to at most one common nonzero component. Thus, $l \geq m + 1$. Hence, we can take $m$ mutually distinct nonzero components, say $T'_1, T'_2, \ldots, T'_m$, such that each component contains a vertex, labeling as $v'_1, v'_2, \ldots, v'_m$, respectively, such that, for each $i$, $Y[v'_i] \neq 0$ and $v'_i \in T'_i$ adjacent to $v_i$.

For each $i$ ($1 \leq i \leq m$), from Lemmas 2.5 and 2.6 we have $Y[v_i] = 0$. Thus,

$$L[T'_i]Y[T'_i] = \lambda Y[T'_i],$$

for each $i$.

Write $T - C_Y(T, Y)$ as $T'$ for simplicity. Let $Y_1 = [Y[T'_1]^T \ 0 \ 0 \ \cdots \ 0]^T$, where the zeros are appended so that $(L[T'_1] - \lambda I)Y_1 = 0$. One can readily verify that $L[v_1, T'_1]Y_1 = L[v_1, T'_1]Y[T'_1] = L[v_1, v'_1]Y_1[v'_1] \neq 0$, since the vector $L[v_1, T'_1]$ has exactly one nonzero coordinate $L[v_1, v'_1]$ and $Y_1[v'_1] \neq 0$. Thus, applying Lemma 2.4,

$$m_{T - C_Y(T, Y)}(k\lambda) = m_{T}[k\lambda] + 1.$$

Further, let $Y_2 = [Y[T'_2]^T \ 0 \ 0 \ \cdots \ 0]^T$, where the zeros are appended so that $(L[T'_2 \cup \{v_1\}] - \lambda I)Y_2 = 0$. Thus, by a similar discussion, $L[v_2, T'_2 \cup \{v_1\}]Y_2 = L[v_2, v'_2]Y_2[v'_2] \neq 0$. Applying Lemma 2.4 again, we have

$$m_{T[T'_2 \cup \{v_1, v_2\}]}(k\lambda) = m_{T[T'_2 \cup \{v_1\}]}(k\lambda) + 1 = m_{T'}(k\lambda) + 2.$$

Using the above operation repeatedly, we have

$$m_{L}(k\lambda) = m_{L[T \cup C_Y]}(k\lambda) = m_{L[T \cup \{v_1, \ldots, v_m\}]}(k\lambda) + 1 = \cdots = m_{L[T \cup \{v_1\}]}(k\lambda) + m - 1 = m_{L[T']}^{n-1}(k\lambda) + m.
\]

Thus,

$$m_{L[T']}^{n-1}(k\lambda) = m_{L}(k\lambda) - m.$$

Consequently,

$$\lambda_{k-1-m}(L[T']) < k \lambda \text{ and } \lambda_{k-m}(L[T']) = k \lambda,$$
since \( k\lambda \) is an eigenvalue of \( L[T'] \) by Lemma 2.3.

Without loss of generality, suppose \( M_2(C_T; k\lambda) = \{T_1^*, T_2^*, \ldots, T_q^*\} \). For each \( i(i = 1, 2, \ldots, q) \), let \( t_i = m_{L[T_i^*]}(k\lambda) \). From Lemma 2.3, \( |C(T_i^*, Y[T_i^*])| = m_{L[T_i^*]}(k\lambda) = t_i \) holds for each \( i \). Then

\[
\sum_{i=1}^{q} |C(T_i^*, Y[T_i^*])| = \sum_{i=1}^{q} t_i =: t^*.
\]

Hence,

\[
|C(T, Y)| = |C(T - C_Y, Y[T - C_Y])| + |C_Y| = \sum_{i=1}^{d} |C(T_i, Y[T_i])| + m = \sum_{i=1}^{d} |C(T_i^*, Y[T_i^*])| + m = t^* + m.
\]

On the other hand, note that \( S(T') = \bigcup_{i=1}^{p+1} S(T_i) \) and each eigenvalue corresponding to the component being of \( M_1(C_Y; k\lambda) \) or \( M_2(C_Y; k\lambda) \) is no less than \( k\lambda \), then \( t + t^* = k - 1 - m \). Hence,

\[
|C(T, Y)| = t^* + m = k - 1 - t. \tag{5}
\]

Putting Theorem 3.4 together with Lemma 2.3, we can give the characterization for the structure of the trees with any possible cardinality of the characteristic set with respect to its \( k\)-maximal vector.

**Theorem 3.5.** Let \( T \) be a tree with Laplacian matrix \( L \), and let \( Y \) be an arbitrary \( k\)-maximal vector of \( T \). Then \(|C(T, Y)| = k - 1\) if and only if every coordinate of \( Y \) different from zero, or \( M_1(C_Y; k\lambda) = \emptyset \) holds for each \( v \in C(T, Y) \).

**Theorem 3.6.** Let \( T \) be a tree with Laplacian matrix \( L \), and let \( Y \) be an arbitrary \( k\)-maximal vector of \( T \). Suppose \( t \) is an arbitrary integer with \( 1 \leq t \leq k - 2 \). Then \(|C(T, Y)| = k - 1 - t\) if and only if \( C_Y(T, Y) \neq \emptyset \) and \( M_4(C_Y; k\lambda) = \{ T_i : i = 1, 2, \ldots, p(p \geq 1) \} \), where \( t = \bigoplus_{i=1}^{p} m_{L[T_i]}(k\lambda) \).

Below we give an example to show the occurrence of each of the case described in the above theorems.

**Example 3.7.** Let \( T \) be a tree on \( n = 2m + 4p + q + 1 \) (\( p + q \geq 2 \)) vertices obtained from a star on \( m + p + q + 1 \) vertices by appending \( m \) pendent edges to \( m \) pendent vertices and \( p \) paths with length 3 to other \( p \) pendent vertices, respectively, see Figure 3.1.

By a little calculation, we have \( k\lambda(T) = 1 \) with multiplicity \( p + q - 1 \), \( M_1(u; 1) = \emptyset \), \( M_2(u; 1) = \{ T[v_{4p+2m+i}] : i = 1, 2, \ldots, q \} \), \( M_4(u; 1) = \{ T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}] : i = \ldots \} \).
The Characteristic Set With Respect to $k$-Maximal Vectors of a Tree

1, 2, ..., $p$, and $M_4(u; 1) = \{T[v_{4p+i}, v_{4p+m+i}] : i = 1, 2, \ldots, m\}$, where $k = m + p + 2$ and $T[S]$ is the subgraph of $T$ induced by its vertex subset $S$. Then the $k$-vector $Y$ has the following partitioned form:

$$Y = [t_1 X_1^T \; t_2 X_2^T \; \cdots \; t_p X_p^T \; W_1^T \; \cdots \; W_m^T \; s_1 \cdots s_q \; 0],$$

where $X_i = [-1 \; 1 \; 0 \; 1]^T$ for $i = 1, 2, \ldots, p$, $W_l = [0 \; 0]^T$ for $l = 1, 2, \ldots, m$, and each $t_i$ (or $s_j$) is real such that the vertex $u$ satisfies eigencondition.

One can see that, for each $k$-vector $Y$, $t_i X_i$ is either zero or non-zero containing both positive entries and negative entries for each $i$, $s_j$ is either zero, or positive, or negative for each $j$, and $W_l = 0$ for each $l$, which is consistent with Lemma [23]. On the other hand, from the partitioned form of $Y$, $u \in C(T, Y)$ if some $t_i$ (or $s_j$) is nonzero. Moreover, we have $|C(T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}], Y([v_i, v_{p+i}, v_{2p+i}, v_{3p+i}])| = |C(T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}], t_i X_i)|$ is either 1 or 0 according to the real $t_i$ is nonzero or not. Thus,

$$|C(T, Y)| = 1 + \sum_{i=1}^{p} |C(L[T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}], Y[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}])|$$

$$\leq 1 + p,$$

from which we have that $Y$ is $k$-maximal if $t_i \neq 0$ for each $i$, which is consistent with Lemma [23]. One can also find that, with respect to any $k$-maximal vector $Y$, $|C(T, Y)| = 1 + p$ regardless of the choice of the integer $m$. Hence, $|C(T, Y)| = 1 + p = k - 1$ if $m = 0$ (in such a case $M_4(u; 1) = \emptyset$), and $|C(T, Y)| = 1 + p \leq k - 2$ otherwise, which is consistent with Lemma [3.1]

Moreover, if $Y$ is $k$-maximal, then $C_T(T, Y) = \{u\} \cup \{v_{2p+i} : i = 1, 2, \ldots, p\}$, and $M_1(C_T; 1) = M_3(C_T; 1) = \emptyset$, $M_2(C_T; 1) = \{T[v_{4p+2m+i}] : i = 1, 2, \ldots, q\} \cup \{T[v_{3p+j}] : j = 1, 2, \ldots, p\} \cup \{T[v_i, v_{p+i}] : i = 1, 2, \ldots, m\}$, which is consistent with Theorems [5.4] and [3.6].
REFERENCES