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BICYCLIC DIGRAPHS WITH EXTREMAL SKEW ENERGY∗

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Abstract. Let $\overrightarrow{G}$ be a digraph and $S(\overrightarrow{G})$ be the skew-adjacency matrix of $\overrightarrow{G}$. The skew energy of $\overrightarrow{G}$ is the sum of the absolute values of eigenvalues of $S(\overrightarrow{G})$. In this paper, the bicyclic digraphs with minimal and maximal skew energy are determined.

Key words. Bicyclic digraph, Skew-adjacency matrix, Extremal skew energy.

AMS subject classifications. 05C05, 05C50, 15A03.

1. Introduction. Let $G$ be a simple undirected graph of order $n$ with vertex set $V(G) = \{1, \ldots, n\}$ and $\overrightarrow{G}$ be an orientation of $G$. The skew-adjacency matrix of $\overrightarrow{G}$ is the $n \times n$ matrix $S(\overrightarrow{G}) = [s_{i,j}]$, $s_{i,j} = 1$ and $s_{j,i} = -1$ if $i \rightarrow j$ is an arc of $\overrightarrow{G}$, and $s_{i,j} = s_{j,i} = 0$ otherwise. Since $S(\overrightarrow{G})$ is a real skew symmetric matrix, all eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of $S(\overrightarrow{G})$ are pure imaginary numbers, the singular values of $S(\overrightarrow{G})$ are just the absolute values $\{|\lambda_1|, \ldots, |\lambda_n|\}$. So the energy of $S(\overrightarrow{G})$ defined as the sum of singular values of $S(\overrightarrow{G})$ [12] is the sum of the absolute values of its eigenvalues. For convenience, the energy of $S(\overrightarrow{G})$ is called skew energy of $G$ ([1]), and denoted by $E_s(\overrightarrow{G})$.

Energy has close links to chemistry (see, for instance, [6]). Since the concept of the energy of simple undirected graphs was introduced by Gutman in [5], there has been lots of research papers on this topic. For a survey, we refer to Section 7 in [3] and references therein. Denote, as usual, the $n$-vertex path and cycle by $P_n$ and $C_n$, respectively. For the extremal energy of bicyclic graphs, let $G(n)$ be the class of bicyclic graphs with $n$ vertices and containing no disjoint odd cycles of lengths $k$ and $\ell$ with $k + \ell = 2 \pmod{4}$. Let $S^f_{\ell}$ be the graph obtained by connecting $n - \ell$ pendant vertices to a vertex of $C_{\ell}$, $S^{3,3}_n$ be the graph formed by joining $n - 4$ pendant vertices to a vertex of degree three of the $K_4 - e$ (see Fig. 1.1), and $S^{4,4}_n$ be the graph formed by joining $n - 5$ pendant vertices to a vertex of degree three of the complete bipartite graph $K_{2,3}$. Let $B_n$ be the class of all bipartite bicyclic graphs of order $n$ that are not the graph obtained from two cycles $C_a$ and $C_b$ ($a, b \geq 10$ and $a = b = 2 \pmod{4}$) joined by an edge. Zhang and Zhou [14] showed that $S^{3,3}_n$ is the graph with

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minimal energy in $G(n)$. In [10], Li et al. showed that $P_{n}^{6,6}$ is the graph with maximal energy in $B_n$. Additional results on the energy of bicyclic graphs can be found in [4, 7, 11, 13]. The skew energy was first introduced by C. Adiga et al. in [1]. Some properties of the skew energy of a digraph are given in [1]. A connected graph with $n$ vertices and $n + 1$ edges is called a bicyclic graph. In this paper, we are interested in studying the bicyclic graphs with extremal skew energy.

The rest of this paper is organized as follows. In Section 2, the bicyclic digraphs of each order $n$ with minimal skew energy are determined. In Section 3, the bicyclic digraphs of each order $n$ with maximal skew energy are determined.

2. Bicyclic graphs with minimal skew energy. Let $G$ be a graph. A linear subgraph $L$ of $G$ is a disjoint union of some edges and some cycles in $G$ ([2]). A $k$-matching $M$ in $G$ is a disjoint union of $k$-edges. If $2k$ is the order of $G$, then a $k$-matching of $G$ is called a perfect matching of $G$. The number of $k$-matchings of graph $G$ is denoted by $m(G, k)$. If $C$ is an even cycle of $G$, then we say $C$ is evenly oriented relative to an orientation $\odot G$ of $G$ if it has an even number of edges oriented in the direction of the routing. Otherwise $C$ is oddly oriented. We call a linear subgraph $L$ of $G$ evenly linear if $L$ contains no cycle with odd length and denote by $\mathcal{EL}_i(G)$ (or $\mathcal{EL}_i$ for short) the set of all evenly linear subgraphs of $G$ with $i$ vertices. For a linear subgraph $L \in \mathcal{EL}_i$, denote by $p_e(L)$ (resp., $p_o(L)$) the number of evenly (resp., oddly) oriented cycles in $L$ relative to $\odot G$. Denote the characteristic polynomial of $S(\odot G)$ by

$$P_S(\odot G; x) = \det(xI - S(\odot G)) = \sum_{i=0}^{n} b_i x^{n-i}. \tag{1}$$

Then $b_0 = 1$, $b_2$ is the number of edges of $G$, all $b_i \geq 0$ and $b_i = 0$ for all odd $i$. 

**Fig. 1.1.** Graphs $S_n^{3,3}$, $P_n^{4,4}$ and their orientations.
We have the following results.

**Lemma 2.1.** (8) Let $\vec{G}$ be an orientation of a graph $G$. Then
\[
b_i(\vec{G}) = \sum_{L \in \mathcal{E}_i} (-2)^{p_L} 2^{p_0(L)}.
\]

**Lemma 2.2.** (8) Let $e = uv$ be an edge of $G$ that is on no even cycle of $G$. Then
\[
(2.1) \quad P_S(\vec{G};x) = P_S(\vec{G} - e; x) + P_S(\vec{G} - u - v; x).
\]

By equating the coefficients of polynomials in Eq. (2.1), we have
\[
(2.2) \quad b_{2k}(\vec{G}) = b_{2k}(\vec{G} - e) + b_{2k-2}(\vec{G} - u - v).
\]

Furthermore, if $e = uv$ is a pendant edge with pendant vertex $v$, then
\[
(2.3) \quad b_{2k}(\vec{G}) = b_{2k}(\vec{G} - v) + b_{2k-2}(\vec{G} - u - v).
\]

For any orientation of a graph that does not contain any even cycle (in particular, a tree or a unicyclic non-bipartite graph), $b_{2k}(\vec{G}) = m(\vec{G}, k)$ by Lemma 2.1.

In [9], the skew energy of $\vec{G}$ is expressed as the following integral formula:
\[
E_s(\vec{G}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2} \ln(1 + \sum_{k=1}^{[n/2]} b_{2k} t^{2k}) dt.
\]

Thus $E_s(\vec{G})$ is an increasing function of $b_{2k}(\vec{G})$, $k = 0, 1, \ldots, [n/2]$. Consequently, if $\vec{G}_1$ and $\vec{G}_2$ are oriented graphs of $G_1$ and $G_2$, respectively, for which
\[
(2.4) \quad b_{2k}(\vec{G}_1) \geq b_{2k}(\vec{G}_2)
\]
for all $[n/2] \geq k \geq 0$, then
\[
(2.5) \quad E_s(\vec{G}_1) \geq E_s(\vec{G}_2).
\]

Equality in (2.5) is attained only if (2.4) is an equality for all $[n/2] \geq k \geq 0$. If the inequalities (2.4) hold for all $k$, then we write $G_1 \succeq G_2$ or $G_2 \preceq G_1$. If $G_1 \succeq G_2$, but not $G_2 \succeq G_1$, then we write $G_1 \succ G_2$.

Let $\vec{G}$ be an orientation of a graph $G$. Let $W$ be a subset of $V(G)$ and $\overline{W} = V(G) \setminus W$. The orientation $\overline{\vec{G}}$ of $G$ obtained from $\vec{G}$ by reversing the orientations...
of all arcs between \( \overrightarrow{W} \) and \( W \) is said to be obtained from \( \overrightarrow{G} \) by a switching with respect to \( W \). Moreover, two orientations \( \overrightarrow{G} \) and \( \overrightarrow{G}' \) of a graph \( G \) are said to be switching-equivalent if \( \overrightarrow{G}' \) can be obtained from \( \overrightarrow{G} \) by a sequence of switchings. As noted in [1], since the skew adjacency matrices obtained by a switching are similar, their spectra and hence skew energies are equal.

It is easy to verify that up to switching equivalence there are just two orientations of a cycle \( C \): (1) Just one edge on the cycle has the opposite orientation to that of others, we denote this orientation by \( + \). (2) All edges on the cycle \( C \) have the same orientation, we denote this orientation by \( - \). So if a cycle is of even length and oddly oriented, then it is equivalent to the orientation \( + \); if a cycle is of even length and evenly oriented, then it is equivalent to the orientation \( - \). The skew energy of a directed tree is the same as the energy of its underlying tree ([1]). So by switching equivalence, for a unicyclic digraph or bicyclic digraph, we only need to consider the orientations of cycles.

Let \( C_x, C_y \) be two cycles in bicyclic graph \( G \) with \( t \) \( (t \geq 0) \) common vertices. If \( t \leq 1 \), then \( G \) contains exactly two cycles. If \( t \geq 2 \), then \( G \) contains exactly three cycles. The third cycle is denoted by \( G_z \), where \( z = x + y - 2t + 2 \). Without loss of generality, assume that \( x \leq y \leq z \).

For convenience, we denote by \( G^+ \) (resp., \( G^- \)) the unicyclic graph on which the orientation of a cycle is of orientation \( + \) (resp., \( - \)), and denote by \( G^* \) the unicyclic graph on which the orientation of a cycle is of arbitrary orientation \( * \). If \( t \leq 1 \), we denote by \( G^{a,b} \) the bicyclic graph on which cycle \( C_x \) is of orientation \( a \) and cycle \( C_y \) is of orientation \( b \), where \( a, b \in \{+, -, *\} \). If \( t \geq 2 \), we denote by \( G^{a,b,c} \) the bicyclic graph on which \( C_x \) is of orientation \( a \), \( C_y \) is of orientation \( b \), \( C_z \) is of orientation \( c \), where \( a, b, c \in \{+, -, *\} \). See examples in Fig. 2.2.

For the \( k \)-matching number of a graph \( G \), we have the following.

**Lemma 2.3.** Let \( e = uv \) be an edge of \( G \). Then

(i) \( m(G, k) = m(G - e, k) + m(G - u - v, k - 1) \).

(ii) If \( G \) is a forest, then \( m(G, k) \leq m(P_n, k), k \geq 1 \).

(iii) If \( H \) is a subgraph of \( G \), then \( m(H, k) \leq m(G, k), k \geq 1 \). Moreover, if \( H \) is a proper subgraph of \( G \), then the inequality is strict.

We define \( m(G, 0) = 1 \) and \( m(G, k) = 0 \) for \( k > \frac{n}{2} \).

In [9], the authors discussed the unicyclic digraph with extremal skew energy and established the following.

**Lemma 2.4.** (1) Among all unicyclic digraphs on \( n \) vertices, \( \overrightarrow{G}^3 \) has the
minimal skew energy and $S_n^3$ has the second minimal skew energy for $n \geq 6$; both $S_n^3$ and $S_n^4$ have the minimal skew energy, $S_n^3$ has the second minimal skew energy for $n = 5$; $C^3$ has the minimal skew energy, $S_n^4$ has the second minimal skew energy for $n = 4$.

(2) Among all orientations of unicyclic graphs, $P_n^4$ is the unique directed graph with maximal skew energy.

For bicyclic digraphs, we have the following.

**Lemma 2.5.** Let $\overrightarrow{G}$ be a bicyclic digraph of order $n \geq 8$, $G \neq S_n^{3,3}$. Then $\overrightarrow{G} \succ (S_n^{3,3})^{+,+,+}$.

**Proof.** We prove the statement by induction on $n$. By Lemma 2.1, the characteristic polynomials of $S((S_n^{3,3})^{+,+,+})$, $S(S_n^3)$ and $S(S_n^4)$ are:

$$P_S((S_n^{3,3})^{+,+,+}) = x^{n-4}(x^4 + (n + 1)x^2 + 2(n - 4)),$$

$$P_S(S_n^3) = x^{n-4}(x^4 + nx^2 + (n - 3)),$$

$$P_S(S_n^4) = x^{n-4}(x^4 + nx^2 + 2(n - 4)).$$

It suffices to prove that $b^4(\overrightarrow{G}) > 2(n - 4)$ for $\overrightarrow{G} \neq (S_n^{3,3})^{+,+,+}$.

Let $n = 8$. 

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**Fig. 2.1.** Examples for orientation representations of bicyclic digraphs.

$G^+ = 2$, $G^+ = 2$, $G^+ = 2$,

$G^+_1$, $G^+_2$, $G^+_3$,
Case 1.1. \( t \leq 1 \).

Subcase 1.1.1. \( x = y = 4 \). Then we can choose an edge \( e = uv \) on some \( C_4 \) such that \( G - u - v \) is connected. By Lemma 2.1 we have

\[
b_{4}(G^{*,*}) \geq m(G, 2) - 4 \\
= m(G - e, 2) + m(G - u - v, 1) - 4 \\
\geq m(G - e, 2) + 6 - 4 \\
> m(P_6, 2) + 2 \quad (P_6 \text{ is a proper subgraph of } G - e) \\
= 8 = b_{4}((S_{3}^{3,3})^{*,*,*}).
\]

Subcase 1.1.2. Either \( x \) or \( y \) is 4. Without loss of generality, suppose that \( x = 4 \). Chose an edge \( e = uv \) on \( C_y \) such that \( G - u - v \) has at least 4 edges. By Lemma 2.1

\[
b_{4}(G^{*,*}) \geq m(G, 2) - 2 \\
= m(G - e, 2) + m(G - u - v, 1) - 2 \\
\geq m(G - e, 2) + 4 - 2 \\
> m(S_{4}^{3,2}) + 2 \quad (S_{4}^{3} \text{ is a proper subgraph of } G - e) \\
> 8 = b_{4}((S_{3}^{3,3})^{*,*,*}).
\]

Subcase 1.1.3. Neither \( x \) nor \( y \) is 4. Then \( x = y = 3 \) or \( x = 3, y = 5 \). We can chose an edge \( e = uv \) on any cycle such that \( G - u - v \) contains at least 3 edges. By Lemma 2.1 we get

\[
b_{4}(G^{*,*}) = m(G, 2) \\
= m(G - e, 2) + m(G - u - v, 1) \\
\geq m(G - e, 2) + 3 \\
= b_{4}(G - e) + 3 \quad (G - e \text{ is a unicyclic graph without cycle of length 4}) \\
> b_{4}(S_{3}^{3}) + 3 \quad \text{(by Lemma 2.1)} \\
= 8 = b_{4}((S_{3}^{3,3})^{*,*,*}).
\]

Case 1.2. \( t \geq 2 \).

Subcase 1.2.1. Each cycle is of length 4. Then \( t = 3 \) and there are 3 vertices outside of those cycles, say \( v_1, v_2, v_3 \). Let \( v_1 \) be a pendant vertex of \( G \) and \( u_1 \) be the adjacent vertex of \( v_1 \), \( v_2 \) be a pendant vertex of \( G - v_1 \) and \( u_2 \) be the adjacent vertex of \( v_2 \), \( v_3 \) be a pendant vertex of \( G - v_1 - v_2 \) and \( u_3 \) be the adjacent vertex of \( v_3 \). By Eq. 2.3, we have

\[
b_{4}(G^{*,*}) = b_{4}(G^{*,*} - v_1) + b_{2}(G^{*,*} - u_1 - v_1)
\]
Subcase 1.2.2. There are two cycles of length 4 in $G$, say, $C_x, C_y$, then $t = 2$ and $z = x + y - 2t + 2 = 6$. There are two vertices outside of $G$. Similar to the proof of subcase 1.2.1, we can obtain $b_4(G^{*,*}) > b_4((S_{8}^{3},{3})^{*,*,*})$.

Subcase 1.2.3. There is just one cycle of length 4 in $G$, say, $C_x$. If $C_y = C_3$, then $t = 2$ and there are 3 vertices outside of $G$. Similar to the proof of subcase 1.2.1, we get $b_4(G^{*,*}) > b_4((S_{8}^{3})^{*,*,*,*})$. If $y \geq 5$, we can chose an edge $e = uv$ on $C_y$ such that $m(G - u - v, 1) \geq 6$. Then similar to the proof of subcase 1.1.2, we get that $b_4(G^{*,*}) > b_4((S_{8}^{3},3)^{*,*,*,*})$.

Subcase 1.2.4. $G$ contains no cycle of length 4. Similar to the proof of subcase 1.1.3, the result holds for $n = 8$.

Suppose $n > 8$ and $\overrightarrow{G} = (S_{8}^{3},3)^{*,*,*,*}$ for any bicyclic digraph $G'$ of order $n'$, $n' < n$. Denote by $p$ the number of pendant vertices in $G$.

If $p = 0$, then $\overrightarrow{G}$ has no pendant vertex. Three cases are considered in the following.

Case 2.1. $t = 1$. Let $e = uv$ be an edge on $C_x$ and $u$ is the common vertex of $C_x$ and $C_y$. By Lemmas 2.1 and 2.3 and $m(P_n, 2) = \binom{n-2}{2}$, we have

\[
b_4(G^{*,*}) \geq m(G, 2) - 4
= m(G - e, 2) + m(G - u - v, 1) - 4
= m(P_n^y, 2) + m(P_{n-2} \cup P_{n-y, 1}) - 4
= m(P_n, 2) + m(P_{n-2} \cup P_{n-y, 1}) + n - 8
= \frac{(n-2)(n-3)}{2} + n - 4 + n - 8 > 2(n - 4)
\]

since $\frac{(n-2)(n-3)}{2} - 4 = \frac{(n-6)(n+1)+4}{2} > 0$ for $n > 7$.

Case 2.2. $t \geq 2$. Suppose $e = uv$ is an edge on $C_x$ and $u$ is the common vertex of $C_x$ and $C_y$. By Lemmas 2.1 and 2.3

\[
b_4(G^{*,*}) \geq m(G, 2) - 6
= m(G - e, 2) + m(G - u - v, 1) - 6
= m(P_n^y, 2) + n - 3 - 6
\]
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\[ = m(P_n, 2) + m(P_{y-2} \cup P_{n-y}, 1) + n - 9 \]
\[ = \frac{(n-2)(n-3)}{2} + n - 4 + n - 9 > 2(n-4) \]

since \( \frac{(n-2)(n-3)}{2} - 5 = \frac{(n-6)(n+1)+2}{2} > 0 \) for \( n > 7 \).

**Case 2.3.** \( t = 0 \). Suppose that \( C_x \) and \( C_y \) are joined by a path of length \( a \), \( n - 8 \geq a \geq 0 \). Let \( e = uv \) be an edge on \( C_x \), where \( u \) is the vertex of degree 3. Similar to the proof of case 2.1, we obtain \( b_4(G^{x,*}) > 2(n-4) \). Therefore, \( b_4(G^{x,*}) \geq b_4((S_n^{3,3})^{*,*,-}) \) for \( p = 0 \).

Let \( p \geq 1 \) and \( v \) be a pendant vertex of \( G \) with corresponding unique edge \( uv \). Since \( G^{x,*} - u - v \) has at least 3 edges, by Eq. (2.2) and the induction hypothesis,

\[ b_4(G^{x,*}) = b_4(G^{x,*} - v) + b_2(G^{x,*} - u - v) \]
\[ > b_4((S_n^{3,3})^{*,*,-}) + 3 \]
\[ = 2(n - 1 - 4) + 3 > 2(n - 4) = b_4((S_n^{3,3})^{*,*,*}). \]

For \( n = 7 \), similar to the proof of Lemma 2.5 for \( n = 8 \), both \( (S_n^{4,4})^{*,*,-,-} \) and \( (S_n^{3,3})^{*,*,*,-} \) have the minimal skew energy. Since

\[ P_5((S_6^{4,4})^{*,*,-,-}) = x^{n-4}(x^n + (n + 1)x^2 + 3(n - 5)), \]
\[ b_4((S_6^{4,4})^{*,*,-,-}) = 3 \text{ and } b_4((S_5^{4,4})^{*,*,-,-}) = 0. \] In a similar way to the proof of Lemma 2.5 for \( n = 8 \), we can get that \( (S_n^{4,4})^{*,*,-,-} \) has the minimal skew energy for \( n = 6 \) and \( (S_5^{4,4})^{*,*,-,-} \) has the minimal skew energy for \( n = 5 \).

By Lemma 2.5 and the above statements, we obtain the following.

**Theorem 2.6.** Among all bicyclic digraphs of order \( n \), \( (S_n^{3,3})^{*,*,*,-} \) has the minimal skew energy for \( n \geq 8 \); both \( (S_n^{3,3})^{*,*,*,-} \) and \( (S_n^{4,4})^{*,*,-,-} \) have the minimal skew energy for \( n = 7 \); \( (S_n^{4,4})^{*,*,-,-} \) has the minimal skew energy for \( n = 5, 6 \).

**3. Bicyclic digraphs with maximal skew energy.** For the path, by Lemmas 2.1 and 2.3 we can easily get the following statements.

**Lemma 3.1.** Let \( \overrightarrow{P_n} \) be a forest of order \( n \). Then \( \overrightarrow{P_n} \preceq \overrightarrow{P_n} \). Equality holds if and only if \( F_n = P_n \).

Since the skew energy of a directed forest is the same as the energy of its underlying forest, by [10], we have the following.
Lemma 3.2.

\[ \vec{P}_n \supset \vec{P}_2 \bigcup \vec{P}_{n-2} \supset \vec{P}_4 \bigcup \vec{P}_{n-4} \supset \cdots \supset \vec{P}_{2k} \bigcup \vec{P}_{n-2k} \supset \vec{P}_{2k+1} \bigcup \vec{P}_{n-2k-1} \]
\[ \times \vec{P}_{2k-1} \bigcup \vec{P}_{n-2k+1} \supset \cdots \supset \vec{P}_3 \bigcup \vec{P}_{n-3} \supset \vec{P}_1 \bigcup \vec{P}_{n-1}. \]

Lemma 3.3. For any bicyclic graph \( G \) with \( t \leq 1 \), \( G^{*,*} \preceq G^{+,+} \).

Proof. If two cycles are of odd length, then by Lemma 2.1, for any orientation of \( G \), \( b_{2k}(\vec{G}) = m(G, k) \), for all \( 0 \leq k \leq [\frac{t}{2}] \). Thus \( G^{*,*} = G^{+,+} \). If there is exactly one cycle of even length in \( G \), say, \( C_x \), then

\[ b_{2k}(G^{*,*}) = m(G, k) - 2m(G - C_x, k - \frac{x}{2}) \leq b_{2k}(G^{+,+}) = m(G, k) + 2m(G - C_x, k - \frac{x}{2}). \]

If both \( x \) and \( y \) are even, then

\[ b_{2k}(G^{*,*}) = m(G, k) - 2m(G - C_x, k - \frac{x}{2}) - 2m(G - C_y, k - \frac{y}{2}) \]
\[ + 4m(G - C_x - C_y, k - \frac{x+y}{2}) \leq b_{2k}(G^{+,+}) = m(G, k) \]
\[ \pm 2m(G - C_x, k - \frac{x}{2}) \mp 2m(G - C_y, k - \frac{y}{2}) \]
\[ = 2m(G - C_x - C_y, k - \frac{x+y}{2}) \leq b_{2k}(G^{+,+}) = m(G, k) \]
\[ + 2m(G - C_x, k - \frac{x}{2}) + 2m(G - C_y, k - \frac{y}{2}) \]
\[ + 4m(G - C_x - C_y, k - \frac{x+y}{2}). \]

Lemma 3.4. Let \( \vec{G} \) be a bicyclic digraph of order \( n \) with \( t \leq 1 \), \( \vec{G} \neq (P_n^{+,+})^{*,+} \). Then \( \vec{G} \prec (P_n^{+,+})^{*,+} \) for \( n \geq 8 \).

Proof. We divide the proof into two cases.

Case 1. There is at least one cycle of length odd, say, \( C_x \).

(i) \( t = 1 \), we can choose an edge \( e = uv \) on \( C_x \) such that \( u \) is the common vertex of two cycles. Obviously, \( \vec{G} - e \) is a unicyclic graph and \( \vec{G} - u - v \) is a forest.

By Eq. (2.2), Lemmas 2.3 and 2.4 we have

\[ b_{2k}(\vec{G}) = b_{2k}(\vec{G} - e) + b_{2k-2}(\vec{G} - u - v) \]
\[ < b_{2k}(P_n^{+,+}) + b_{2k-2}(\vec{P}_2 \bigcup \vec{P}_n) \quad \text{(by Lemmas 2.4 and 3.2)} \]
\[ = m(P_n^{+,+}, k) + 3m(P_{n-4}, k - 2) + m(P_{n-4}, k - 1) \]
\[ < m(P_n^{+,+}, k) + m(P_{n-4}^{+,+}, k - 1) + 5m(P_{n-4}^{+,+}, k - 2) + 4m(P_{n-8}, k - 4). \]
By Eq. (2.2), Lemmas 2.1, 2.3 and 2.4, we have

\( b_{2k}(G^{+,-}) = b_{2k}((P_n^{4,4})^{+,+}). \)

(ii) \( t = 0 \). We can choose an edge \( e = uv \) on \( C_x \) such that \( u \) is a vertex in a path which connects \( C_x \) and \( C_y \). Obviously, \( \overrightarrow{G} - e \) is a unicyclic graph and \( \overrightarrow{G} - u - v \) is the disjoint union of a forest and a unicyclic graph.

Claim 1. \( P_n \cup \overrightarrow{P_{n-a}} < P_2 \cup \overrightarrow{P_{n-2}}, a \neq 2. \)

Proof. By Lemmas 2.4 and 3.2, we have

\[
\begin{aligned}
b_{2k}(P_n \cup \overrightarrow{P_{n-a}}) &< b_{2k}(P_n \cup \overrightarrow{P_{n-2}}) \\
&= m(P_n \cup P_{n-a}, k) + 2m(P_n \cup P_{n-a-4}, k - 2) \\
&< m(P_n \cup P_{n-a}, k) + m(P_n \cup P_{n-a-4}, k - 1) \\
&+ 2m(P_n \cup P_{n-a-4}, k - 2) \\
&< m(P_n \cup P_{n-2}, k) + m(P_n \cup P_{n-2}, k - 1) \\
&+ 2m(P_n \cup P_{n-2}, k - 2) \\
&= b_{2k}(P_2 \cup \overrightarrow{P_{n-2}}). \quad \Box
\end{aligned}
\]

By Eq. (2.2), Lemmas 2.4, 3.2 and 2.4, we have

\[
\begin{aligned}
b_{2k}(G) &= b_{2k}(\overrightarrow{G}) + b_{2k-2}(\overrightarrow{G} - u - v) \\
&< b_{2k}(P_n^+) + b_{2k-2}(P_2 \cup \overrightarrow{P_{n-4}}) \quad \text{(by Claim 1)} \\
&= m(P_n^+, k) + 2m(P_{n-4}, k - 2) + m(P_2 \cup P_{n-4}, k - 1) \\
&+ 2m(P_2 \cup P_{n-4}, k - 2) \\
&\leq m(P_n^+, k) + m(P_2 \cup P_{n-4}, k - 1) + 4m(P_{n-4}, k - 2) \\
&\leq m(P_n^+, k) + m(P_2 \cup P_{n-4}, k - 1) + 4m(P_{n-4}, k - 2) + 4m(P_{n-8}, k - 4) \\
&= b_{2k}((P_n^{4,4})^{+,+}).
\end{aligned}
\]

Case 2. Two cycles are of even lengths.

By Lemma 3.3, we only need to consider \( G^{+,+} \).

(1) \( t = 1 \). We can choose an edge \( e = uv \) in cycle \( C_x \), \( u \) is the common vertex of two cycles. By Lemma 2.4, we have

\[
b_{2k}(G^{+,+}) = m(G, k) + 2m(G - C_x, k - \frac{x}{2}) + 2m(G - C_y, k - \frac{y}{2})
\]
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\[ \leq m(G-e,k) + m(G-u-v,k-1) + 2m(P_{n-4},k-2) \]
\[ + 2m(G-C_y,k-\frac{y}{2}) \]
\[ = b_{2k}(\overrightarrow{G-e}) + m(G-u-v,k-1) + 2m(P_{n-4},k-2) \]
\[ \leq b_{2k}(\overrightarrow{G-e}) + m(P_2 \bigcup P_{n-4},k-1) + 2m(P_{n-4},k-2) \]
\[ < b_{2k}(\overrightarrow{P_n}) + m(P_2 \bigcup P_{n-4},k-1) + 2m(P_{n-4},k-2) \]
\[ = m(P_4^4,k) + m(P_2 \bigcup P_{n-4},k-1) + 4m(P_{n-4},k-2) \]
\[ \leq m(P_4^4,k) + m(P_2 \bigcup P_{n-4},k-1) + 4m(P_{n-4},k-2) + 4m(P_{n-8},k-4) \]
\[ = b_{2k}((P_n^4)^{+,+}). \]

(ii) \( t = 0 \). Choose an edge \( e = uv \) on the path which connects \( C_x \) and \( C_y \). Assume that one of the two components of \( G-e \) is of order \( j \). By Eq. (2.2), we get

\[ b_{2k}(G^{+,+}) = b_{2k}(G^{+,+}-e) + b_{2k-2}(G^{+,+}-u-v) \]
\[ < b_{2k}(\overrightarrow{P_j^4} \bigcup \overrightarrow{P_{n-j}^4}) + b_{2k-2}(\overrightarrow{P_{n-j}^4} \bigcup \overrightarrow{P_{n-j-1}^4}) \]
\[ ( \text{or} < b_{2k}(\overrightarrow{P_j^4} \bigcup \overrightarrow{P_{n-j}^4}) + b_{2k-2}(\overrightarrow{P_3^4} \bigcup \overrightarrow{P_{n-5}^4})) \]
\[ = b_{2k}((P_n^4)^{+,+}). \]

**Lemma 3.5.** Let \( \overrightarrow{G} \) be a bicyclic digraph of order \( n \) with \( t \geq 2 \). Then \( \overrightarrow{G} \prec (P_n^4)^{+,+} \) for \( n \geq 8 \).

**Proof.** We prove statement by dividing three cases.

**Case 1.** \( x = y = z = 4 \). Then \( t = 3 \). If both \( C_x \) and \( C_y \) are oddly oriented, then \( C_z \) must be evenly oriented. We can choose an edge \( e = uv \) such that \( G-u-v \) is disconnected. Without loss of generality, we assume that \( e \) is on \( C_y \).

\[ b_{2k}(G^{+,+}) = m(G,k) + 2m(G-C_x,k-2) + 2m(G-C_y,k-2) \]
\[ - 2m(G-C_z,k-2) \]
\[ \leq m(G-e,k) + m(G-u-v,k-1) + 2m(G-C_x,k-2) \]
\[ + 2m(G-C_y,k-2) \]
\[ \leq b_{2k}(\overrightarrow{G-e}) + m(P_2 \bigcup P_{n-4},k-1) + 2m(P_{n-4},k-2) \]
\[ < b_{2k}(\overrightarrow{P_n^4}) + m(P_2 \bigcup P_{n-4},k-1) + 2m(P_{n-4},k-2) \]
\[ = m(P_4^4,k) + m(P_2 \bigcup P_{n-4},k-1) + 4m(P_{n-4},k-2) \]
\[ \leq m(P_4^4,k) + m(P_2 \bigcup P_{n-4},k-1) + 4m(P_{n-4},k-2) \]

\[ \Box \]
we can prove that
\[ b_{2k}(P^{4,4}_n) = 4m(P_{n-8}, k - 4) \]
\[ = b_{2k}((P^{4,4}_n)^{+,+}). \]

If either \( C_x \) or \( C_y \) is oddly oriented, then \( C_z \) must be oddly oriented. Similarly, we can prove that \( b_{2k}(G^{+,+,+}) < b_{2k}((P^{4,4}_n)^{+,+}) \) or \( b_{2k}(G^{+,+,+}) < b_{2k}((P^{4,4}_n)^{+,+}). \)

If both \( C_x \) and \( C_y \) are evenly oriented, then \( C_z \) is also evenly oriented.

\[
\begin{align*}
b_{2k}(G^{-,-,-}) &= m(G, k) - 2m(G - C_x, k - 2) - 2m(G - C_y, k - 2) - 2m(G - C_z, k - 2) \\
&
\leq m(G, k) + 2m(G - C_x, k - 2) + 2m(G - C_y, k - 2) - 2m(G - C_z, k - 2) \\
&= b_{2k}(G^{+,+,+}) < b_{2k}((P^{4,4}_n)^{+,+}).
\end{align*}
\]

**Case 2.** \( x = y = 4, z \neq 4 \). Then \( t = 2 \) and \( z = 6 \). If both \( C_x \) and \( C_y \) are oddly oriented, then \( C_z \) is oddly oriented. Since \( n \geq 8 \), we can choose an edge \( e = uv \) such that \( G - u - v \) is disconnected and \( u \) is one of the common vertices between \( C_x \) and \( C_y \). Without loss of generality, we suppose that \( e \) is on \( C_y \). Then both \( G - C_y \) and \( G - C_z \) are also disconnected. Note that \( G - C_x = G - e - C_x \), by Lemma \(2.1\) we get

\[
\begin{align*}
b_{2k}(G^{+,+,+}) &= m(G, k) + 2m(G - C_x, k - 2) + 2m(G - C_y, k - 2) \\
&+ 2m(G - C_z, k - 3) \\
&\leq m(G - e, k) + m(G - u - v, k - 1) + 2m(G - e - C_x, k - 2) \\
&+ 2m(P_2 \bigcup P_{n-6}, k - 2) + 2m(P_2 \bigcup P_{n-8}, k - 3) \\
&\leq b_{2k}(G^{-,-}) + m(P_2 \bigcup P_{n-4}, k - 1) \\
&+ 2m(P_2 \bigcup P_{n-6}, k - 2) + 2m(P_2 \bigcup P_{n-8}, k - 3) \\
&< b_{2k}((P^{4,4}_n) + m(P_2 \bigcup P_{n-4}, k - 1) \\
&+ 2m(P_2 \bigcup P_{n-6}, k - 2) + 2m(P_2 \bigcup P_{n-8}, k - 3) \\
&= m(P^{4,4}_n, k) + m(P_2 \bigcup P_{n-4}, k - 1) + 2m(P_{n-4}, k - 2) \\
&+ 2m(P_2 \bigcup P_{n-6}, k - 2) + 2m(P_2 \bigcup P_{n-8}, k - 3) \\
&\leq m(P^{4,4}_n, k) + m(P_2 \bigcup P^{4,4}_{n-4}, k - 1) + 4m(P^{4,4}_{n-4}, k - 2) \\
&+ 4m(P_{n-8}, k - 4) \\
&= b_{2k}((P^{4,4}_n)^{+,+}).
\end{align*}
\]

If either \( C_x \) or \( C_y \) is oddly oriented, then \( C_z \) is evenly oriented. By Lemma
Claim 2 follows immediately.

If both $C_x$ and $C_y$ are evenly oriented, then $C_z$ is oddly oriented and

$$b_{2k}(G^+,-,-) \leq b_{2k}(G^+,+,+) < b_{2k}((P^4_n)^+,+), \text{ or } b_{2k}(G^-,+,+) \leq b_{2k}(G^+,+,+) < b_{2k}((P^4_n)^+,+).$$

Case 3. There aren’t two cycles with length 4. Since there is at least one cycle of even length in $G$, without loss of generality, we assume that $C_x$ is a cycle of minimal even length.

Subcase 3.1. $t$ is even.

Subcase 3.1.1. $y$ is even and both $C_x$ and $C_y$ are oddly oriented. Then $y,z > x \geq 4$ and $C_z$ is oddly oriented. Let $e = uv$ be an edge on $C_y$ and $u$ is the common vertex of $C_x$ and $C_y$.

Claim 2. $m(P_{n-2}, k-1) \geq m(P_{n-4}, k-2) \geq \cdots \geq m(P_{n-2\ell}, k-\ell)$.

Proof. By Lemma 2.3 we get

$$m(P_{n-2}, k-1) = m(P_{n-3}, k-1) + m(P_{n-4}, k-2) = m(P_{n-3}, k-1) + m(P_{n-5}, k-2) + m(P_{n-6}, k-3) = \sum_{i=1}^{\ell-1} m(P_{n-(2i+1)}, k-i) + m(P_{n-2\ell}, k-\ell).$$

Claim 2 follows immediately. □

By Lemma 2.1 and Claim 2, we get

$$b_{2k}(G^+,+,+) = m(G, k) + 2m(G - C_x, k-2) + 2m(G - C_y, k - \frac{y}{2}) + 2m(G - C_x, k - \frac{y}{2})$$

$$\leq m(G - e, k) + m(G - u - v, k-1) + 2m(G - e - C_x, k-2) + 4m(P_{n-6}, k-3)$$

$$\leq b_{2k}(G^+e\rightarrow) + m(P_{n-2}, k-1) + 4m(P_{n-6}, k-3)$$

$$< b_{2k}(P^4_n) + m(P_{n-2}, k-1) + 4m(P_{n-8} \cup P_2, k-3) + 4m(P_{n-9}, k-4)$$

$$= m(P^4_n, k) + 2m(P_{n-4}, k-2) + m(P_2 \cup P_{n-4}, k-1) + m(\cup P_{n-5}, k-2) + 4m(P_{n-8}, k-3) + 4m(P_{n-9}, k-4)$$

$$\leq m(P^4_n, k) + m(P_2 \cup P^4_{n-4}, k-1) + 4m(P_{n-4}, k-2)$$
2.1, we get

\[ +4m(P_{n-s} \cup P_2, k - 3) + 4m(P_{n-s}, k - 4) = b_{2k}((P_n^4)^{+,+}). \]

If either \( C_x \) or \( C_y \) is oddly oriented, then \( C_z \) is evenly oriented. By Lemma 2.1, \( b_{2k}(G^{+,+,+}) < b_{2k}((P_n^4)^{+,+}) \), or \( b_{2k}(G^{+,+,+}) < b_{2k}((P_n^4)^{+,+}) \).

If both \( C_x \) and \( C_y \) are evenly oriented, then \( C_z \) is oddly oriented and

\[ b_{2k}(G^{+,+,+}) < b_{2k}((P_n^4)^{+,+}). \]

**Subcase 3.1.** \( y \) is odd. Then \( z \) is also odd. We can choose an edge \( e = uv \) on \( C_y \) such that \( u \) is the common vertex and \( v \) is not the common vertex. By Lemma 2.1, we get

\[ b_{2k}(G^{+,+,+}) \leq b_{2k}(G^{+,+,+}) = b_{2k}(G - e^+) + b_{2k-2}(G - u - v) \]

\[ < b_{2k}(P_n^4) + b_{2k-2}(P_{n-2}) \]

\[ = m(P_n^4, k) + 2m(P_{n-4}, k - 2) + m(P_{n-2}, k - 1) \]

\[ < m(P_n^4, k) + m(P_2 \cup P_{n-4}, k - 1) + m(P_{n-5}, k - 2) + 2m(P_{n-4}, k - 2) \]

\[ \leq m(P_n^4, k) + m(P_2 \cup P_{n-4}, k - 1) + 4m(P_{n-4}, k - 2) \]

\[ + 4m(P_{n-8}, k - 4) \]

\[ = b_{2k}((P_n^4)^{+,+}). \]

**Subcase 3.2.** \( t \) is odd.

**Subcase 3.2.1.** \( y \) is even. Then \( y > 4 \). If both \( C_x \) and \( C_y \) are oddly oriented, then \( z \) is even and \( C_z \) is evenly oriented. Let \( e = uv \) be an edge on \( C_y \) and \( u \) is the common vertex between \( C_x \) and \( C_y \). Then

\[ b_{2k}(G^{+,+,+}) = m(G, k) + 2m(G - C_z, k - 2) + 2m(G - C_y, k - \frac{y}{2}) \]

\[ - 2m(G - C_z, k - \frac{y}{2}) \]

\[ \leq m(G - e, k) + m(G - u - v, k - 1) + 2m(G - C_x, k - 2) \]

\[ + 2m(P_{n-6}, k - 3) \]

\[ \leq b_{2k}(G - e^+) + m(P_{n-2}, k - 1) + 2m(P_{n-6}, k - 3) \]

\[ < b_{2k}(P_n^4) + m(P_{n-2}, k - 1) + 2m(P_{n-6}, k - 3) \]

\[ = m(P_n^4, k) + 2m(P_{n-4}, k - 2) + m(P_2 \cup P_{n-4}, k - 1) + m(P_{n-5}, k - 2) \]

\[ + 2m(P_{n-6}, k - 3) \]
If either $C_x$ or $C_y$ is oddly oriented, then $C_z$ is oddly oriented. Similar to the above proof, $b_{2k}(G^{-,-,+}) < b_{2k}((P^4_n)^{+,+})$ or $b_{2k}(G^{-,+,-}) < b_{2k}((P^4_n)^{+,+})$.

If both $C_x$ and $C_y$ are evenly oriented, then $C_z$ is evenly oriented, so

$$b_{2k}(G^{-,-,-}) \leq b_{2k}(G^{+,+,-}) < b_{2k}((P^4_n)^{+,+}).$$

Subcase 3.2.2. $y$ is odd. Then $z$ is odd too. Similar to the proof of subcase 3.1.2, we obtain $b_{2k}(G^{-,*,*}) < b_{2k}((P^4_n)^{+,+})$.

Combining all those cases above, we complete the proof.

By identifying two vertices of two cycles with length 4, we get a graph $G^{4,4}_n$. For $n = 6, 7$, similar to the proofs of Lemmas 3.4, 3.5, we obtain that the following graphs have the maximal skew energy.

![Graph](image)

**Fig. 3.1.** The maximal skew energy graph $(P^4_6)^{+,+}$ for $n = 6$ and $(G^4_7)^{+,+}$ for $n = 7$.

By Lemmas 3.4 and 3.5 we obtain the following statement.

**Theorem 3.6.** Among all bicyclic digraphs with order $n \geq 8$, $(P^4_n)^{+,+}$ has the maximal skew energy; $(G^4_7)^{+,+}$ has the maximal skew energy for $n = 7$; $(P^4_6)^{+,+}$ has the maximal skew energy for $n = 6$.

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