Representations for the Drazin inverse of block cyclic matrices

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Abstract. A formula for the Drazin inverse of a block $k$-cyclic ($k \geq 2$) matrix $A$ with nonzeros only in blocks $A_{i,i+1}$, for $i = 1, \ldots, k$ (mod $k$) is presented in terms of the Drazin inverse of a smaller order product of the nonzero blocks of $A$, namely $B_i = A_{i,i+1} \cdots A_{i-1,i}$ for some $i$. Bounds on the index of $A$ in terms of the minimum and maximum indices of these $B_i$ are derived. Illustrative examples and special cases are given.

Key words. Drazin inverse, Block cyclic matrix, Index.

AMS subject classifications. 15A09.

1. Introduction. We consider $k$-cyclic ($k \geq 2$) block real or complex matrices of the form

$$A = \begin{bmatrix}
0 & A_{12} & 0 & \cdots & 0 \\
0 & 0 & A_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{k-1,k} \\
A_{k1} & 0 & 0 & \cdots & 0
\end{bmatrix},$$

(1.1)

where $A_{12}, \ldots, A_{k1}$ are block submatrices and the diagonal zero blocks are square. It is easily verified that for any matrix $A$ of the form (1.1), the Moore-Penrose inverse $A^\dagger$ of $A$ is given by

$$A^\dagger = \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{k1}^\dagger \\
A_{12}^\dagger & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & A_{23}^\dagger & \cdots & 0 & 0 \\
0 & 0 & \cdots & A_{k-1,k}^\dagger & 0
\end{bmatrix},$$

(1.2)

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where \( A^\dagger_{ij} \) denotes the Moore-Penrose inverse of the block submatrix \( A_{ij} \). Note that if each of the blocks \( A_{ij} \) is square and invertible, then (1.2) gives the formula for the usual inverse \( A^{-1} \) of \( A \). We present a block formula for another type of generalized inverse, the Drazin inverse, of matrices of the form (1.1). Unlike the Moore-Penrose inverse, the Drazin inverse is defined only for square matrices.

Let \( A \) be a real or complex square matrix. The Drazin inverse of \( A \) is the unique matrix \( A_D \) satisfying

\[
\begin{align*}
AA_D &= A_D A \\
A_D AA_D &= A_D \\
A^{q+1}A_D &= A^q,
\end{align*}
\]

where \( q = \text{index } A \), the smallest nonnegative integer \( q \) such that \( \text{rank } A^{q+1} = \text{rank } A^q \). If \( \text{index } A = 0 \), then \( A \) is nonsingular and \( A_D = A^{-1} \). If \( \text{index } A = 1 \), then \( A_D = A^\# \), the group inverse of \( A \). See [1], [2], [6] and references therein for applications of the Drazin inverse.

**Theorem 1.1.** [2, Theorem 7.2.3] Let \( A \) be a square matrix with index \( A = q \). If \( p \) is a nonnegative integer and \( X \) is a matrix satisfying \( XAX = X \), \( AX =XA \), and \( A^{q+1}X = A^p \), then \( p \geq q \) and \( X = A_D \).

The problem of finding explicit representations for the Drazin inverse of a general 2 \( \times \) 2 block matrix of the form

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\]

in terms of its blocks was posed by Campbell and Meyer in [2], and special cases of this problem were the focus of several recent papers, including [3]–[10], [13], [14] and [15]. In [4] and [13], representations for 2 \( \times \) 2 block matrices matrices of the form (1.6) with \( A_{11} \) and \( A_{22} \) being square zero diagonal blocks were presented. Such block matrices were called bipartite (or 2-cyclic), and in this article, we extend the results given in [4] to general block k-cyclic matrices as defined in (1.1).

**2. Drazin inverse formula for block cyclic matrices.** Let \( A \) be a block k-cyclic matrix of the form given in (1.1). For our Drazin inverse formula we introduce some notation that is also used in writing powers of \( A \). For \( i = 2, \ldots, k-1 \), let \( B_i \) be the square matrix defined by

\[
B_i = A_{i,i+1} \cdots A_{k-1,k} A_{k1} A_{12} \cdots A_{i-1,i},
\]

with \( B_1 = A_{12} A_{23} \cdots A_{k-1,k} A_{k1} \) and \( B_k = A_{k1} A_{12} \cdots A_{k-1,k} \), i.e., subscripts are taken mod \( k \). For ease of notation, we define the matrix product

\[
A_{i\rightarrow j} := A_{i,i+1} A_{i+1,i+2} \cdots A_{j-1,j},
\]
for \( j \neq i \). Whenever it arises, we use the convention \( A_{i \rightarrow i} = I \), an identity matrix. For example, if \( k = 4 \) then

\[
A = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{pmatrix}
\]

and by Lemma 2.1 \( B_{3} = A_{34}A_{41}A_{12}A_{23} \). Observe that \( B_{i} = A_{i \rightarrow j}A_{j \rightarrow i} \), for any \( j \in \{1, \ldots, k\} \setminus \{i\} \).

**Lemma 2.2.** For all \( i \neq j \), \( B_{i}A_{i \rightarrow j} = A_{i \rightarrow j}B_{j} \).

**Proof.** \( B_{i}A_{i \rightarrow j} = (A_{i \rightarrow j}A_{j \rightarrow i})^{k}A_{i \rightarrow j} = A_{i \rightarrow j}A_{j \rightarrow i}(A_{i \rightarrow j}A_{j \rightarrow i})^{k-1}A_{i \rightarrow j} = A_{i \rightarrow j}(A_{j \rightarrow i}A_{i \rightarrow j})^{k} = A_{i \rightarrow j}B_{j}^{k} \). 

**Lemma 2.3.** For all \( i \neq j \), \( B_{i}^{D}A_{i \rightarrow j} = A_{i \rightarrow j}B_{j}^{D} \). Hence, if \( \ell \neq i, j \) satisfies \( A_{i \rightarrow j} = A_{i \rightarrow \ell}A_{\ell \rightarrow j} \), then \( B_{i}^{D}A_{i \rightarrow j} = A_{i \rightarrow j}B_{j}^{D} = A_{i \rightarrow \ell}B_{\ell}^{D}A_{\ell \rightarrow j} \).
Proof. \( B_i^D A_{i\rightarrow j} = (A_{i\rightarrow j}A_{j\rightarrow i})^D A_{i\rightarrow j} = A_{i\rightarrow j}(A_{j\rightarrow i}A_{i\rightarrow j})^D = A_{i\rightarrow j}B_j^D \), where the second equality is due to [4, Lemma 2.4]. □

With the above notation, we now give a formula for the Drazin inverse of a \( k \)-cyclic matrix \( A \) given by (1.1).

**Theorem 2.4.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined as in (2.1) and \( A_{i\rightarrow j} \) defined in (2.2). Then, for all \( i = 1, \ldots, k \),

\[
A_D = \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{1\rightarrow i}B_i^D A_{i\rightarrow k} \\
A_{2\rightarrow i}B_i^D A_{i\rightarrow 1} & 0 & \cdots & 0 & 0 \\
0 & A_{3\rightarrow i}B_i^D A_{i\rightarrow 2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k\rightarrow i}B_i^D A_{i\rightarrow k-1} & 0
\end{bmatrix}.
\]

Moreover, if \( \text{index } B_i = s_i \), then \( \text{index } A \leq ks_i + k - 1 \).

**Proof.** Denote the matrix on the right hand side of (2.6) by \( X \). Performing block multiplication gives

\[
AX = \begin{bmatrix}
A_{12}A_{2\rightarrow 1}B_i^D A_{i\rightarrow 1} & 0 & \cdots & 0 \\
0 & A_{23}A_{3\rightarrow 1}B_i^D A_{i\rightarrow 2} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & A_{k1}A_{1\rightarrow i}B_i^D A_{i\rightarrow k}
\end{bmatrix}
\]

\[
A_{12}A_{2\rightarrow 1}A_{1\rightarrow 1}B_1^D & 0 & \cdots & 0 \\
0 & A_{23}A_{3\rightarrow 1}A_{1\rightarrow 2}B_1^D & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & A_{k1}A_{1\rightarrow i}A_{i\rightarrow k}B_k^D
\]

(by Lemma 2.3)

\[
= \begin{bmatrix}
B_1B_1^D & 0 & \cdots & 0 \\
0 & B_2B_2^D & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_kB_k^D
\end{bmatrix}.
\]
and by using Lemma 2.3 again

\[ XA = \begin{bmatrix} A_{1\rightarrow i} B_i^D A_{i\rightarrow k} A_{k+1} & 0 & \cdots & 0 \\ 0 & A_{2\rightarrow i} B_i^D A_{i\rightarrow k+1} & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{k\rightarrow i} B_i^D A_{i\rightarrow k+1} \end{bmatrix} \]

\[ = \begin{bmatrix} B_i^D A_{1\rightarrow i} A_{i\rightarrow k+1} & 0 & \cdots & 0 \\ 0 & B_i^D A_{2\rightarrow i} A_{i\rightarrow k+1} & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_i^D A_{k\rightarrow i} A_{i\rightarrow k+1} \end{bmatrix} \]

\[ = \begin{bmatrix} B_i^D B_1 & 0 & \cdots & 0 \\ 0 & B_i^D B_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_i^D B_k \end{bmatrix} \begin{bmatrix} B_i^D & 0 & \cdots & 0 \\ 0 & B_i^D & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_i^D \end{bmatrix} \]

\[ = AX, \]

since \( B_i^D B_i = B_i B_i^D \) by (1.3). Also, block-multiplying \( X \) with \( AX \) gives

\[ XAX = X(AX) \]

\[ = \begin{bmatrix} 0 & 0 & \cdots & 0 & A_{1\rightarrow k} B_i^D B_k B_i^D \\ A_{2\rightarrow i} B_i^D B_i^D & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{k\rightarrow k-1} B_i^D B_{k-1} B_i^D \end{bmatrix} \]

\[ = X, \text{ by Lemma 2.3 and since } B_i^D B_i B_i^D = B_i^D \text{ by (1.4).} \]

Let \( i \) be any integer in \( \{1, \ldots, k\} \) and suppose that index \( B_i = s_i = s \). Then using (2.3) and Lemma 2.2

\[ A^{k+k} X = A^{k(s+1)} X \]

\[ = \begin{bmatrix} 0 & 0 & \cdots & 0 & A_{1\rightarrow i} B_i^{s+1} B_i^D A_{i\rightarrow k} \\ A_{2\rightarrow i} B_i^{s+1} B_i^D A_{i\rightarrow k+1} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & A_{k\rightarrow i} B_i^{s+1} B_i^D A_{i\rightarrow k-1} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & A_{k\rightarrow i} B_i^{s+1} B_i^D A_{i\rightarrow k-1} & 0 \end{bmatrix} . \]
Since index $B_i = s$, it follows by (1.5) that $B_i^{s+1}B_i^D = B_i^s$. Thus, using Lemma 2.2 and $A_{\ell \rightarrow i}, A_{i \rightarrow j} = A_{\ell \rightarrow j}$ for $\ell \neq j$,

$$A^{ks+k}X = \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{1 \rightarrow k}B_k^s \\
A_2^{s-1}B_1^s & 0 & \cdots & 0 & 0 \\
0 & A_3^{\rightarrow 2}B_2^s & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k \rightarrow k-1}B_{k-1}^s & 0 \\
\end{bmatrix} = A^{ks+k-1},$$

from (2.5) by using Lemma 2.2. By Theorem 1.1, index $A \leq ks+k-1$ and $X = A^D$. \[Q.E.D.\]

Thus, the Drazin inverse of a $k$-cyclic matrix is reduced to calculating the Drazin inverse of the smallest order Drazin inverse of any of the matrix products $B_i$.

**Corollary 2.5.** If $A$ of the form in (1.1) is nonnegative and has at least one $B_i^D \geq 0$, then $A^D$ is nonnegative.

The following example illustrates Theorem 2.4 and Corollary 2.5.

**Example 2.6.** Let

$$A = \begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
0 & A_{12} & 0 \\
0 & 0 & A_{23} \\
A_{31} & 0 & 0 \\
\end{bmatrix}.$$  

Then $B_1 = A_{12}A_{23}A_{31} = 1, B_2 = A_{23}A_{31}A_{12} = \frac{1}{2}J_2$ (where $J_2$ is $2 \times 2$ all ones matrix) and $B_3 = A_{31}A_{12}A_{23} = 1$. Note that index $B_i = 0$ and $B_i^D = B_i^{-1} = 1$.

Using Theorem 2.4,

$$A^D = \begin{bmatrix}
0 & 0 & B_1^D A_{12}A_{23} \\
A_2^{s-1}B_1^s & 0 & 0 \\
0 & A_3^{\rightarrow 2}B_2^s & 0 \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & \frac{1}{2} \frac{1}{2} & 0 \\
\end{bmatrix} = A^2.$$  

In fact, rank $A = \text{rank} A^2$, hence $A^D = A^\# = A^2$ agreeing with Theorem 2.2 in [11].

### 3. Index of A in relation to the indices of the block products. With $A$ as in (1.1), for $j \geq 0$, by (2.3) and (2.4),

\[\text{(3.1)} \quad \text{rank } A^{kj} = \text{rank } B_1^j + \text{rank } B_2^j + \cdots + \text{rank } B_k^j\]

\[\text{(3.2)} \quad \text{rank } A^{kj+1} = \text{rank } B_1^jA_{12} + \text{rank } B_2^jA_{23} + \cdots + \text{rank } B_k^jA_{ki}.\]
The following rank inequality is used throughout the proof of Lemma 3.2 and can be found in standard linear algebra texts (see, e.g., [12 page 13]).

LEMMA 3.1. (Frobenius Inequality) If $U$ is $m \times n$, $V$ is $n \times p$ and $W$ is $p \times q$, then

$$\text{rank } UV + \text{rank } V = \text{rank } V + \text{rank } UVW.$$ 

LEMMA 3.2. Let $A$ be as in (1.1) with associated matrices $B_i$ defined in (2.1), and let $s = \text{index } B_i \geq 1$ for some $i \in \{1, \ldots, k\}$. Then $\text{rank } A^{k_s-k+1} < \text{rank } A^{k_s-k}$.

Proof. Let $s = \text{index } B_i$ for some $i \in \{1, \ldots, k\}$. From (3.2),

$$\text{rank } A^{k_s-k+1} = \text{rank } A^{k(s-1)+1} = \text{rank } B_i^{s-1} A_{12} + \text{rank } B_i^{s-2} A_{23} + \text{rank } B_i^{s-3} A_{34} + \cdots + \text{rank } B_i^{k_s-1} A_{k1},$$

where the terms can be reordered as

$$\text{rank } B_i^{s-1} A_{i,i+1} + \text{rank } B_i^{s-1} A_{i+1,i+2} + \cdots + \text{rank } B_i^{s-1} A_{k1} + \text{rank } B_i^{s-1} A_{12} + \cdots$$

(3.3) + \text{rank } B_i^{k_s-1} A_{i-1,1}.

Using Lemma 2.2 the first two terms in the expression in (3.3) can be written as

$$\text{rank } A_i B_i^{s-1} + \text{rank } B_i^{s-1} A_{i+1,i+2},$$

and using the Frobenius inequality (Lemma 3.1),

$$\text{rank } B_i^{s-1} A_{i,i+1} + \text{rank } B_i^{s-1} A_{i+1,i+2} \leq \text{rank } B_i^{s-1} + \text{rank } A_i B_i^{s-1} A_{i+1,i+2} = \text{rank } B_i^{s-1} + \text{rank } B_i^{s-1} A_{i-1,1},$$

where the equality is again due to Lemma 2.2. Thus,

$$\text{rank } A^{k_s-k+1} \leq \text{rank } B_i^{s-1} + \text{rank } B_i^{s-1} A_{i-1,1} \leq \text{rank } B_i^{s-1} + \text{rank } B_i^{s-1} A_{i-1,1} + \text{rank } B_i^{s-1} A_{i-1,1} + \text{rank } A_i B_i^{s-1} A_{i+1,i+2}.$$

(3.4)

Applying Lemma 2.2 and the Frobenius inequality again to the second and third terms on the right-hand side of the inequality in (3.3) gives

$$\text{rank } B_i^{s-1} A_{i-1,1} + \text{rank } B_i^{s-1} A_{i+1,i+2} + \text{rank } B_i^{s-1} A_{i+2,i+3} + \cdots$$

Continuing in this manner gives

$$\text{rank } A^{k_s-k+1} \leq \text{rank } B_i^{s-1} + \text{rank } B_i^{s-1} + \cdots + \text{rank } B_i^{s-1} + \text{rank } A_i B_i^{s-1} A_{i-1,1}.$$
Using Lemma 2.2, the last term on the righthand side of the inequality in (3.5) becomes
\[
\text{rank } B_i^{s-1} A_{i-1} A_{i-1} = \text{rank } B_i^{s-1} B_i = \text{rank } B_i^{s-1},
\]
since index $B_i = s$. Thus,
\[
\text{rank } A^{k_s-k+1} < \text{rank } B_i^{s-1} + \text{rank } B_i^{s-1} + \cdots \text{rank } B_i^{s-1} + \text{rank } B_i^{s-1} + \cdots
\]
\[
= \text{rank } A^{k(s-1)} = \text{rank } A^{k_s-k},
\]
where the equality follows from (3.1).

**Theorem 3.3.** Let $A$ be as in (1.1) with associated matrices $B_i$ defined in (2.1). Then, the following statements hold.

(i) If index $B_i = 0$ for all $i = 1, \ldots, k$, then $A$ is nonsingular and index $A = 0$.

(ii) If index $B_i = s_i \geq 1$ for some $i \in \{1, \ldots, k\}$, then index $A \geq k_s - k + 1$.

**Proof.** The first statement follows immediately from (2.3) and (3.1). For the second statement, let index $B_i = s_i \geq 1$ for some $i \in \{1, \ldots, k\}$. Then $\text{rank } A^{k_s-k+1} < \text{rank } A^{k_s-k}$, by Lemma 3.2. From the strict inequality, index $A \geq k_s - k + 1$.

The next result follows immediately from Theorem 3.3(ii).

**Corollary 3.4.** Let $A$ be as in (1.1) with associated matrices $B_i$ defined in (2.1). If index $A \leq 1$, then index $B_i \leq 1$ for all $i = 1, \ldots, k$. That is, if the group inverse $A^\#$ exists, then the group inverses $B_i^\#$ exist for all $i = 1, \ldots, k$.

Note however that the converse to Corollary 3.4 is false (see, e.g., [4, Example 4.3]).

**Remark 3.5.** If $A$ of the form (1.1) is nonnegative and all matrices with the same $+, 0$ sign pattern as $A$ that have index 1 have at least one $B_i^\#$ nonnegative, then these group inverses are nonnegative (Corollary 2.5) and $A$ is conditionally $S^2GI$ in the notation of Zhou et al. [15].

**Corollary 3.6.** Let $A$ be as in (1.1) with associated matrices $B_i$ defined in (2.1), and let $s = \min_{1 \leq i \leq k} \text{index } B_i$ and $s' = \max_{1 \leq i \leq k} \text{index } B_i > 0$. Then $k s' - k + 1 \leq \text{index } A \leq k s + k - 1$. If $s' = 0$, then index $A = 0$.

Corollary 3.6 leads to a result about the indices of $B_i$ that is of independent interest.

**Theorem 3.7.** Let $A$ be as in (1.1) with associated matrices $B_i$ defined in (2.1), and let $s_\ell = \text{index } B_\ell$ for $\ell \in \{1, \ldots, k\}$. Then $|s_i - s_j| \leq 1$ for all $i, j \in \{1, \ldots, k\}$.
Proof. Let \( s = \min_{1 \leq i \leq k} \text{index } B_i \) and \( s' = \max_{1 \leq i \leq k} \text{index } B_i \), and suppose that \( s' = s + t \) where \( t \geq 0 \). By Corollary 3.6

\[
k(s + t) - k + 1 \leq \text{index } A \leq ks + k - 1.
\]

It follows that

\[
k(s + t) - k + 1 \leq ks + k - 1,
\]

or equivalently,

\[
k(t - 2) + 2 \leq 0.
\]

As \( k \geq 2 \), the inequality above is possible only if \( t \leq 1 \). Thus, \( s' - s = t \leq 1 \) and \( |\text{index } B_i - \text{index } B_j| \leq 1 \) for all \( i, j \).

The next result gives tight bounds on \( \text{index } A \) in terms of the minimum index of the block products \( B_i \). The proof is immediate from Corollary 3.6 and Theorem 3.7.

**Theorem 3.8.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1), and let \( s = \min_{1 \leq i \leq k} \text{index } B_i \). Then, exactly one of the following holds:

(i) \( \text{index } B_i = s \) for all \( i = 1, \ldots, k \), or

(ii) \( \text{index } B_i = s + 1 \) for some \( i = 1, \ldots, k \).

If (i) holds, then \( ks - k + 1 \leq \text{index } A \leq ks + k - 1 \). If (ii) holds, then \( ks + 1 \leq \text{index } A \leq ks + k - 1 \).

The above result generalizes bounds found in [4, Section 3] and shows that if \( k = 2 \) and (ii) holds, then \( \text{index } A = 2s + 1 \).

We now give examples that illustrate Theorem 3.8.

**Example 3.9.** Let \( A \) be the matrix in Example 2.6. Using the notation in Theorem 3.8, \( s = 0 = \text{index } B_1 = \text{index } B_3 \) and \( \text{index } B_2 = 1 = s + 1 \). Applying the result with \( k = 3 \) gives the bounds \( 1 \leq \text{index } A \leq 2 \). Since \( \text{rank } A = \text{rank } A^2 \), \( \text{index } A = 1 = ks + 1 \), which is the lower bound of Theorem 3.8 case (ii).

**Example 3.10.** Let

\[
A = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & A_{12} & 0 \\
0 & 0 & A_{23} \\
A_{31} & 0 & 0
\end{bmatrix}.
\]
Then $B_1 = 3, B_2 = \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$ and $B_3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Note that index $B_1 = 0$ and $B_1^{-1} = \frac{1}{3}$. Using Theorem 2.4, $A^D = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \end{bmatrix}$.

Using the notation in Theorem 3.8, $s = 0 = \text{index } B_1$ and index $B_2 = \text{index } B_3 = 1 = s + 1$. Applying the theorem with $k = 3$ gives the bounds $1 \leq \text{index } A \leq 2$. It can be computed that index $A = 2 = ks + k - 1$, which is the upper bound of Theorem 3.8 case (ii).

**Example 3.11.** Let 

$$A = \begin{bmatrix} 0 & B & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ I & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $B$ is a square matrix and $I$ is an identity matrix of the same order as $B$. Note that $B_i = B$ for all $i$. Suppose that index $B = s$. Then index $A = ks$, the midpoint of the interval $[ks - k + 1, ks + k - 1]$ in Theorem 3.8 case (i), and from Theorem 2.4

$$A^D = \begin{bmatrix} 0 & 0 & \cdots & 0 & B^D B \\ B^D & 0 & \cdots & 0 & 0 \\ 0 & B^D B & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B^D B & 0 \end{bmatrix}.$$
EXAMPLE 3.12. Let

\[
A = \begin{bmatrix}
0 & F & 0 & \cdots & 0 \\
0 & 0 & F & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & F \\
F & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

where \(F\) is a square matrix. Then index \(A = \text{index } F\) and \(B_i = F^k\) for \(i = 1, \ldots, k\). Setting index \(A = \ell\) and index \(B_i = s\) gives \(s = \left\lceil \frac{\ell}{k} \right\rceil\). Thus, index \(A\) can take any value in the interval \([k s - k + 1, k s]\), which is half the range given in Theorem 3.8 case (i).

Examples 3.11 and 3.12 have \(B_i\), and thus index \(B_i\), the same for all \(i\). The following result determines index \(A\) in this case, and the necessary and sufficient conditions reduce to the result of [3] Theorem 3.5] for \(k = 2\).

THEOREM 3.13. Let \(A\) be a block \(k\)-cyclic matrix of the form in (1.3) with associated matrices \(B_i\) defined in (2.1), and suppose that \(s = \min \text{index } B_i \geq 1\). Then index \(A = k s\) if and only if

(i) index \(B_i = s\) for all \(i = 1, \ldots, k\), and

(ii) rank \(B_j^* < \text{rank } B_j^{s-1} A_{j-1} \) for some \(j \in \{1, \ldots, k\}\).

If (i) holds, then rank \(B_i^* = \text{rank } B_j^*\) for all \(i, j = 1, \ldots, k\). If (i) holds but (ii) does not hold, then index \(A < k s\).

Proof. Suppose that index \(A = k s\). Then rank \(A^{k s} < \text{rank } A^{k s-1}\). It follows, using (2.3), (2.5) and (6.1), that \(\sum_{i=1}^k \text{rank } B_i^* < \sum_{i=1}^k \text{rank } B_i^{s-1} A_{i-1} A_i\). Thus, rank \(B_j^* < \text{rank } B_j^{s-1} A_{j-1} A_j\) for some \(j \in \{1, \ldots, k\}\), hence (ii) holds. Suppose on the contrary that (i) does not hold. Then, for some \(j \in \{1, \ldots, k\}\), index \(B_j = s + 1\) (by Theorem 3.8). Thus, rank \(B_j^* > \text{rank } B_j^{s+1}\), hence by (2.3) rank \(A^{k s} = \sum_{i=1}^k \text{rank } B_i^* > \sum_{i=1}^k \text{rank } B_i^{s+1} = \text{rank } A^{k(s+1)}\). This implies that rank \(A^{k s} < \text{rank } A^{k s + k}\), so index \(A > k s\), a contradiction. Hence, (i) and (ii) must hold.

For the reverse implication, suppose that (i) and (ii) hold. Then rank \(A^{k s} = \sum_{i=1}^k \text{rank } B_i^* < \sum_{i=1}^k \text{rank } B_i^{s-1} A_{i-1} A_i = \text{rank } A^{k(s-1) + (k-1)} = \text{rank } A^{k s-1}\), where the strict inequality is due to (ii). Thus, index \(A \geq k s\). Note that since rank \(B_i^* \geq \text{rank } B_i^{s+1} A_{i-1} A_i \geq \text{rank } B_i^{s+1}\) and rank \(B_i^* A_{i-1} A_i = \text{rank } A_{i-1} B_i^{s+1}\) (by Lemma 2.2), it follows using (i) that rank \(B_i^{s+1} = \text{rank } B_i^* = \text{rank } B_i^{s+1} A_{i-1} A_i = \text{rank } A_{i-1} B_i^{s+1} = \text{rank } B_i^{s+1}\) for all \(i, j\). Thus, rank \(A^{k s} = \sum_{i=1}^k \text{rank } B_i^* = \sum_{i=1}^k \text{rank } B_i^{s+1} = \text{rank } A^{k s+1}\), using (3.1) and (3.2). Hence, rank \(A^{k s} = \text{rank } A^{k s+1}\), and so index \(A \leq k s\). This
proves that index $A = ks$. The last two statements of the theorem follow from the proof above.

The result of Theorem 3.13 is illustrated by Example 3.11 since rank $B^s_2 < \text{rank } B_2^s A_2^{-1} = \text{rank } B_2^s A$, it follows that rank $A = ks$. Example 3.12 also illustrates Theorem 3.13 since rank $A^{ks} = \text{rank } A^{k(s+1)}$ and rank $F^{ks} < \text{rank } F^{k(s-1)} F^{k-1} = \text{rank } F^{ks-1}$ if and only if index $F = \text{index } A = ks$; otherwise index $A < ks$.

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