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Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1542

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ON THE NUMERICAL RANGES OF THE WEIGHTED SHIFT OPERATORS WITH GEOMETRIC AND HARMONIC WEIGHTS

ADIYASUREN VANDANJAV† AND BATZORIG UNDRAKH‡

Abstract. In this paper, an exact formula for det \((tI_n - (Q_n + Q^*_n))\) is obtained. This formula yields a simple computation of the numerical ranges of the geometric weighted shift operator \(Q_n\) and the harmonic weighted shift operator \(H_n\) for \(n = 3, 4\).

Key words. Numerical range, Weighted shift operators, Geometric weights, Harmonic weights.

AMS subject classifications. 15A60, 47A12.

1. Introduction. The numerical range of an \(n \times n\) matrix \(T\) is defined as the set

\[ W(T) = \{ \langle Tx, x \rangle : \|x\| = 1 \} \]

where \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\) denote the standard inner product and its associated norm in \(\mathbb{C}^n\). It is known that \(W(T)\) is a nonempty convex subset of \(\mathbb{C}\); see for example [3]. The numerical radius \(w(T)\) of a matrix \(T\) is given by

\[ w(T) = \sup \{ |\lambda| : \lambda \in W(T) \} \]

For its other properties, see [3].

A shift matrix

\[
T = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 \\
\alpha_1 & 0 & 0 & 0 & \ldots & 0 \\
0 & \alpha_2 & 0 & 0 & \ldots & 0 \\
0 & 0 & \alpha_3 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & 0 & \alpha_{n-1} & 0
\end{bmatrix},
\]

and a diagonal matrix

\[ U = \text{diag}(1, \exp(i\theta), \exp(2i\theta), \exp(3i\theta), \ldots, \exp((n-1)i\theta)) \]

Received by the editors on May 19, 2012. Accepted for publication on June 24, 2012. Handling Editor: Michael Tsatsomeros.

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satisfy the equation

$$UTU^* = \exp(i\theta)T,$$

and hence, $$\exp(i\theta)W(T) = W(T)$$ for $$0 \leq \theta \leq 2\pi$$. For a shift matrix, the numerical radius $$w(T)$$ is characterized as the maximum root of the characteristic polynomial

$$P(x) = \det \left( xI_n - \frac{1}{2}(T + T^*) \right).$$

In [2], the value

$$M(\theta) = \max\{\Re(z \exp(-i\theta)) : z \in W(T)\}$$

for a matrix $$T$$ is characterized as the maximum eigenvalue of a hermitian matrix

$$\frac{1}{2}(\exp(i\theta)T + \exp(-i\theta)T^*)$$

$$(0 \leq \theta \leq 2\pi)$$. If $$T$$ is a shift matrix, then the numerical range $$W(T)$$ is a closed circular disc with center at the origin, and hence, $$w(T)$$ is the maximum eigenvalue of a hermitian matrix $$(T + T^*)/2$$.

We consider a weighted shift operator $$A$$ on the Hilbert space $$\ell^2(\mathbb{N})$$ defined by

$$A = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & \vdots \\
a_1 & 0 & 0 & 0 & \ldots & \vdots \\
0 & a_2 & 0 & 0 & \ldots & \vdots \\
0 & 0 & a_3 & 0 & \ldots & \vdots \\
\vdots & \vdots & \vdots & \ldots & \ldots & \vdots \\
\vdots & \vdots & \vdots & \ldots & \ldots & \vdots \\
0 & \ldots & \ldots & 0 & q^{n-2} & 0 \\
0 & \ldots & \ldots & 0 & q^{n-1} & 0 \\
\vdots & \vdots & \vdots & \ldots & \ldots & \vdots \\
\vdots & \vdots & \vdots & \ldots & \ldots & \vdots \\
0 & \ldots & \ldots & 0 & q^{n-2} & 0 \\
\end{bmatrix}$$

(1.2)

where $$\{a_n\}$$ is a bounded sequence. The numerical range is also defined for Hilbert space operators. It is known that $$W(A)$$ is a circular disk centered at the origin [5]. In particular, if the weights are geometric $$a_n = q^{n-1}$$ for some $$0 < q < 1$$ and $$n \in \mathbb{N}$$, then the numerical range of $$T_n$$ is closed disc centered at the origin [1]. Furthermore, the authors of [1] found upper and lower bounds for $$w(T)$$. However, we do not use their result and we develop a simple and different method to solve it. Consider the following two finite operators

$$Q_n = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 \\
0 & q & 0 & 0 & \ldots & 0 \\
0 & 0 & q^2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \ldots & \vdots \\
\vdots & \vdots & \vdots & \ldots & \ldots & \vdots \\
0 & \ldots & \ldots & 0 & q^{n-2} & 0 \\
0 & \ldots & \ldots & 0 & q^{n-1} & 0 \\
\vdots & \vdots & \vdots & \ldots & \ldots & \vdots \\
\vdots & \vdots & \vdots & \ldots & \ldots & \vdots \\
0 & \ldots & \ldots & 0 & q^{n-2} & 0 \\
\end{bmatrix}$$

(1.3)
where $0 < q < 1$ and

$$H_n = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{3} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix}.$$  

(1.4)

In this paper, we study the numerical ranges of matrices defined in (2.10) and (1.4). We give a general exact formula for \( \det (tI_n - (Q_n + Q_n^*)) \). Using this exact formula for \( n = 3, 4 \), we verify that \( W(Q_n) \) and \( W(H_n) \) are closed disks centered at the origin.

2. Geometric weights.

**Theorem 2.1.** Let

$$f_m = \det (zI_m - (Q_m + Q_m^*)) .$$

Then we have

$$f_m(z) = z^m \sum_{k=1}^{[\frac{m}{2}]} (-1)^k z^{m-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1-p^{m-2k+i}}{1-p^i} ,$$

(2.2)

where \( p = q^2 \), for \( m \geq 2 \).

**Proof.** Let \( q^2 = p \). Assume that \( f_0(z) = 1, f_1(z) = z \). Then we have \( f_2(z) = z^2 - 1, f_3(z) = z^3 - z(1 + p) \). Expanding on the last row of the matrix (2.1) leads to the recurrence formula

$$f_{k+2}(z) = z f_{k+1}(z) - p^k f_k(z) .$$

(2.3)

Now we prove (2.2) by induction method. We prove the formula (2.2) for the case \( m = 2n \) and the case \( m = 2n + 1 \) can be done in an analogous way: \( m = 2 \) is trivial. Now assume that (2.2) is holds for \( m = 2, 3, \ldots, n, n+1 \) then we prove that for \( m = n+2 \).

$$f_m(z) = z^{2n} + \sum_{k=1}^{n} (-1)^k z^{2n-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1-p^{2n-2k+i}}{1-p^i} ,$$

(2.4)

$$f_{m+1}(z) = z^{2n+1} + \sum_{k=1}^{n} (-1)^k z^{2n+1-2k} p^{k(k-1)} \prod_{i=1}^{k} \frac{1-p^{2n+1-2k+i}}{1-p^i} .$$

(2.5)
Substituting (2.4) and (2.5) into the (2.3), we have

\[ f_{m+2} = z f_{m+1}(z) - p^m f_m(z) \]

\[ = z^{2n+2} + \sum_{k=1}^{n} (\frac{1}{1-p})^k z^{2n+2-2k} p^k(k-1) \prod_{i=1}^{k} \frac{1-p^{2n+1-2k+i}}{1-p^i} \]

\[ -p^{2n} z^{2n} \sum_{k=1}^{n} (\frac{1}{1-p})^k z^{2n-2k} p^k(k-1) \prod_{i=1}^{k} \frac{1-p^{2n-2k+i}}{1-p^i}. \]

On the other hand, we have

\[ (-1)^k z^{2n+2-2k} p^k(k-1) \prod_{i=1}^{k} \frac{1-p^{2n+1-2k+i}}{1-p^i} \]

\[ -p^{2n} (-1)^{k-1} z^{2n-2k+2} p^k(k-2) \sum_{i=1}^{k-1} \frac{1-p^{2n-2k+2+i}}{1-p^i} \]

\[ = (-1)^k z^{2n-2k+2} p^k(k-1) \frac{(1-p^{2n-2k+3})(1-p^{2n-2k+4}) \cdots (1-p^{2n-k+1})}{(1-p)(1-p^2) \cdots (1-p^{n-1})} \]

\[ \cdot \left[ \frac{1-p^{2n-2k+2}}{1-p^k} + p^{2-2k+2n} \right] \]

\[ = (-1)^k z^{2n-2k+2} p^k(k-1) \frac{(1-p^{2n-2k+3})(1-p^{2n-2k+4}) \cdots (1-p^{2n-k+1})(1-p^{2n-k+2})}{(1-p)(1-p^2) \cdots (1-p^{n-1})(1-p^n)} \]

\[ = (-1)^k z^{2n+2-2k} p^k(k-1) \prod_{i=1}^{k} \frac{1-p^{2n+1-2k+i}}{1-p^i}. \]

Also

\[ z^n \cdot p^0 \cdot \frac{1-p^{2n}}{1-p} = z^n z^{2n} = z^n \left( \frac{1-p^{2n}}{1-p} \right) = z^{2n} \frac{1-p^{2n+1}}{1-p}, \]

and

\[ -p^{2n} \cdot (-1)^{n-1} z^0 \cdot p^{n-1} \prod_{i=1}^{n} \frac{1-p^i}{1-p} = -p^{2n} (-1)^{n-1} \cdot p^{n^2-n} = (1+p)^n \cdot p^{n^2+n} \]

\[ = (-1)^{n+1} z^{2n+2-2(n+1)} \cdot p^{n(n+1)} \prod_{i=1}^{n+1} \frac{1-p^i}{1-p^i}. \]

From (2.7), (2.8) and (2.9), it follows

\[ f_{m+2}(z) = z^{2n+2} + \sum_{k=1}^{n+1} (-1)^k z^{2n+2-2k} p^k(k-1) \prod_{i=1}^{k} \frac{1-p^{2n+2-2k+i}}{1-p^i}. \]
Hence, (2.2) is proved. □

Now we give a simple proof of a well known result; see for example [4].

**Theorem 2.2.** Let $S$ be the shift matrix

\[
S = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & 0 & 1 & 0
\end{bmatrix},
\]

(2.10)

then the numerical range of $S$ is a closed disc with centered at origin and $w(S) = \cos \left( \frac{\pi}{n+1} \right)$.

**Proof.** In Theorem 2.1, we set $q = 1$, Then we have

\[
f_n(z) = z^n + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k C_{n-k}^k z^{n-2k}.
\]

(2.11)

Recalling the Chebyshev polynomials of the second kind, $U_n(x)$, we have

\[
U_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k C_{n-k}^k (2x)^{n-2k} = \prod_{k=1}^{n} \left( x - \cos \left( \frac{k\pi}{n+1} \right) \right).
\]

If we substitute $x = \frac{z}{2}$, then we have

\[
U_n(z/2) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k C_{n-k}^k (z/2)^{n-2k} = \prod_{k=1}^{n} \left( \frac{z}{2} - \cos \left( \frac{k\pi}{n+1} \right) \right)
\]

(2.12)

\[
= \frac{1}{2^n} \prod_{k=1}^{n} \left( z - 2 \cos \left( \frac{k\pi}{n+1} \right) \right).
\]

Now from (2.11) and (2.12), it follows

\[
f_n(z) = \frac{1}{2^n} \prod_{k=1}^{n} \left( z - 2 \cos \left( \frac{k\pi}{n+1} \right) \right).
\]

Hence, as we mentioned Section 1 and from [2], we have

\[
w(S) = \cos \left( \frac{\pi}{n+1} \right)
\]
and the numerical range of $S$ is circular disc with centered at origin. 

**Proposition 2.3.** Let $Q_3$ be the operator in $\mathbb{C}^3$ defined by the matrix

$$
Q_3 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & q & 0
\end{bmatrix}, \quad 0 < q < 1.
$$

Then the numerical range of $Q_3$ is a closed disk centered at the origin and the radius is $\sqrt{1+q^2}$, i.e.,

$$
W(Q_3) = \mathbb{D}(0; \sqrt{1+q^2}).
$$

**Proof.** Setting $m = 3$ in (2.2) yields $f_3(z) = z^3 - z(1 + p)$. The maximum root of the equation $f_3(z) = 0$ is $\sqrt{1+q^2}$. Then, as we mentioned above in Section 1 (see [2]), it is easy to see that $W(Q_3) = \mathbb{D}(0; \sqrt{1+q^2})$. 

**Proposition 2.4.** Let $Q_4$ be the operator in $\mathbb{C}^4$ defined by the matrix

$$
Q_4 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & q & 0 & 0 \\
0 & 0 & q^2 & 0
\end{bmatrix}, \quad 0 < q < 1.
$$

Then the numerical range of $Q_4$ is a closed disk centered at the origin and radius is

$$
\frac{1}{2} \sqrt{\frac{1}{2} \left( (1 + q^2 + q^4) + \sqrt{(1 - q^2 + q^4)(1 + 3q^2 + q^4)} \right)}.
$$

i.e.,

$$
W(Q_4) = \mathbb{D}(0; \frac{1}{2} \sqrt{\frac{1}{2} \left( (1 + q^2 + q^4) + \sqrt{(1 - q^2 + q^4)(1 + 3q^2 + q^4)} \right)}).
$$

**Proof.** Setting $m = 4$ in (2.2) yields $f_4(z) = z^4 - z^2(1 + p + p^2) + p^2$. The maximum root of the equation $f_4(z) = 0$ is

$$
\sqrt{\frac{1}{2} \left( (1 + q^2 + q^4) + \sqrt{(1 - q^2 + q^4)(1 + 3q^2 + q^4)} \right)}.
$$

Then, as we mentioned above in Section 1 (see [2]), it is easy to see that

$$
W(Q_4) = \mathbb{D}(0; \frac{1}{2} \sqrt{\frac{1}{2} \left( (1 + q^2 + q^4) + \sqrt{(1 - q^2 + q^4)(1 + 3q^2 + q^4)} \right)}).
$$
3. Harmonic weights. In this section, we find \( W(H_n) \) for \( n = 3, 4 \). We have

\[
H_n + H_n^* = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \frac{1}{2} & 0 & \ldots & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{3}{2} & \ldots & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & \frac{1}{n-1} \\
0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix},
\]

(3.1)

and let

\[
P_n(x) = \det (xI_n - (H_n + H_n^*)).
\]

(3.2)

We can assume that \( P_0(x) = 1, P_1(x) = x \). Then we have

\[
P_2(x) = 4(x^2 - 1), P_3(x) = 9(4x^3 - 5x).
\]

Expanding on the last row of the matrix (3.2) leads to the recurrence formula

\[
P_n(x) = n^2 (xP_{n-1} - P_{n-2}(x)), \quad n \geq 2.
\]

(3.3)

Now we find the numerical range of \( H_n \) for \( n = 3, 4 \) by using the recurrence formula

\[
W(H_3) = \overline{D}(0; \frac{\sqrt{5}}{4}).
\]

(3.5)

**Proof.** In (3.3), we set \( n = 3 \). Then we have \( P_3(x) = 9(4x^3 - 5x) \). The maximum root of the equation \( P_3(x) = 0 \) is \( \frac{\sqrt{5}}{2} \). Then as we mentioned above in Section 1 (see [2]), it is easy to see that \( W(H_3) = \overline{D}(0; \frac{\sqrt{5}}{4}) \). \( \square \)
Proposition 3.2. In $\mathbb{C}^4$, let $H_4$ be the operator defined by the matrix
\[
H_4 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{3}{2} & 0
\end{bmatrix}.
\] (3.6)

Then, the numerical range of $H_4$ is a closed disk centered at the origin with radius equal to $\frac{1}{2} \sqrt{\frac{49 + 5 \sqrt{73}}{72}}$, i.e.,
\[
W(H_4) = \mathbb{D} \left( 0; \frac{1}{2} \sqrt{\frac{49 + 5 \sqrt{73}}{72}} \right). \quad (3.7)
\]

Proof. In (3.3), we set $n = 4$. Then we have $P_4(x) = 16(36x^4 - 49x^2 + 4)$, The maximum root of the equation $P_4(x) = 0$ is $\sqrt{\frac{49 + 5 \sqrt{73}}{72}}$. Then, as we mentioned above in Section 1 (see [2]), it is easy to see that $W(H_4) = \mathbb{D} \left( 0; \frac{1}{2} \sqrt{\frac{49 + 5 \sqrt{73}}{72}} \right)$. \qed

Acknowledgment. We wish to thank everybody who has helped us and especially our family and great thanks for referee.

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