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THE (SIGNLESS) LAPLACIAN SPECTRAL RADII OF C-CYCLIC GRAPHS WITH N VERTICES AND K PENDANT VERTICES

MUHUO LIU†

Abstract. A connected graph is called a c-cyclic graph if it contains n vertices and n + c − 1 edges. Let C(n, k, c) denote the class of connected c-cyclic graphs with n vertices and k pendant vertices. Recently, the unique extremal graph, which has greatest (respectively, signless) Laplacian spectral radius, in C(n, k, c) has been determined for 0 ≤ c ≤ 3, k ≥ 1 and n ≥ 2c + k + 1. In this paper, the unique graph with greatest (respectively, signless) Laplacian spectral radius in C(n, k, c) is determined for c ≥ 0, k ≥ 1 and n ≥ 2c + k + 1.

Key words. (Signless) Laplacian spectral radius, c-Cyclic graph, Pendant vertex.

AMS subject classifications. 05C50, 05C75, 05C05.

1. Introduction. Throughout the paper, G = (V, E) is a connected undirected simple graph with V = {v_1, v_2, ..., v_n}. Let N(v) be the neighbor set of vertex v, and let d(v) be the degree of v. When d(v) = 1, we call v a pendant vertex of G. In the following, we enumerate the degrees of G in non-increasing order, i.e.,

\[d_1 \geq d_2 \geq \cdots \geq d_n,\]

where \(d_i = d(v_i)\).

If G contains n vertices and n + c − 1 edges, then G is called a c-cyclic graph. In particular, G is called a tree, unicyclic graph, bicyclic graph or a tricyclic graph if c = 0, 1, 2 or 3, respectively. In the coming discussion, n and k are two positive integers, and c is a nonnegative integer. Let C(n, k, c) denote the class of connected c-cyclic graphs with n vertices and k pendant vertices.

Let \(P_n\) and \(C_n\) be a path and a cycle on n vertices, respectively. Generally, \(C_3\) is called a triangle and \(C_4\) is called a quadrilateral. Suppose u is a vertex of a graph G. Suppose \(P_s = w_1w_2\cdots w_s\) and \(C_q = v_1v_2\cdots v_qv_1\), where \(w_i \notin V(G)\) for 1 ≤ i ≤ s and \(v_j \notin V(G)\) for 1 ≤ j ≤ q. If we obtain G’ by identifying the vertex u with \(w_1\), then we say that G’ is obtained from G by attaching the path \(P_s\) to u of G. Similarly,
If we obtain $G'$ by identifying the vertex $u$ with $v_1$, then we say that $G'$ is obtained from $G$ by attaching the cycle $C_q$ to $u$ of $G$.

Paths $P_1, \ldots, P_k$ are said to have almost equal lengths if $|l_i - l_j| \leq 1$ for $1 \leq i, j \leq k$. Denote by $F_n(k, C_4^{(1)}, C_3^{(c-1)})$ the unique connected $c$-cyclic graph on $n$ vertices obtained by attaching $t$ quadrilaterals, $c-t$ triangles, and $k$ paths of almost equal lengths, respectively, to a common vertex. Let $F_n(k, C_3^{(c)})$ be the $c$-cyclic graph on $n$ vertices obtained by attaching $k$ paths of almost equal lengths and $c$ triangles, respectively, to a common vertex, and let $F_n(k, C_4^{(c)})$ define the $c$-cyclic graph on $n$ vertices obtained by attaching $k$ paths of almost equal lengths and $c$ quadrilaterals, respectively, to a common vertex. It follows that $F_n(k, C_3^{(c)}) = F_n(k, C_4^{(0)}, C_3^{(c)})$ and $F_n(k, C_4^{(c)}) = F_n(k, C_4^{(c)}, C_3^{(0)})$.

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of $G$, respectively. The Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$ and the signless Laplacian matrix of $G$ is $Q(G) = D(G) + A(G)$. Denote by $\lambda(G)$ and $\mu(G)$, respectively, the Laplacian spectral radius and signless Laplacian spectral radius of $G$. Thus, $\lambda(G)$ and $\mu(G)$ are equal to the largest eigenvalues of $L(G)$ and $Q(G)$, respectively. It is well-known that $Q(G)$ is positive semidefinite and nonnegative, and, when $G$ is connected, it is irreducible [9]. Thus, when $G$ is connected, by the famous Perron-Frobenius Theorem of non-negative matrices (see e.g. [5]), it follows that $\mu(H) < \mu(G)$ holds for any proper subgraph $H$ of $G$.

We call $G$ an extremal graph in $C(n, k, c)$ of first (respectively, second) type if $G$ has greatest signless Laplacian (respectively, Laplacian) spectral radius in $C(n, k, c)$.

Recently, the extremal graphs, which have greatest (signless) Laplacian spectral radii, in $C(n, k, c)$ has been studied [1, 3, 8, 10, 14, 15]. From these recent results, we can conclude that: $F_n(k, C_4^{(0)})$ is the unique extremal tree of $C(n, k, 0)$ [3, 14]; $F_n(k, C_3^{(1)})$ is the unique extremal unicyclic graph in $C(n, k, 1)$ of first type [10, 15] when $n \geq k + 3$ and $F_n(k, C_4^{(1)})$ is the unique extremal unicyclic graph in $C(n, k, 1)$ of second type [3, 10] when $n \geq k + 4$; $F_n(k, C_4^{(2)})$ is the unique extremal bicyclic graph in $C(n, k, 2)$ of first type [1, 11] when $n \geq k + 5$ and $F_n(k, C_4^{(2)})$ is the unique extremal bicyclic graph in $C(n, k, 2)$ of second type [3, 10] when $n \geq k + 7$; $F_n(k, C_3^{(3)})$ is the unique extremal tricyclic graph in $C(n, k, 3)$ of first type [3, 11] when $n \geq k + 7$ and $F_n(k, C_4^{(3)})$ is the unique extremal tricyclic graph in $C(n, k, 3)$ of second type [1] when $n \geq k + 10$.

In this paper, the unique extremal graph in $C(n, k, c)$ of first (respectively, second) type is identified for $c \geq 0$, $k \geq 1$ and $n \geq 2c + k + 1$. Thus, the main results of [1, 3, 4, 8, 10, 11] immediately follow from our results. Our main results can be stated

\begin{itemize}
  \item The (Signless) Laplacian Spectral Radii of $c$-Cyclic Graphs
\end{itemize}
as follows.

**Theorem 1.1.** If \( k \geq 1, c \geq 0 \) and \( n \geq 2c + k + 1 \), then \( F_n(k, C_3^{(c)}) \) is the unique extremal graph in \( C(n, k, c) \) of first type.

**Theorem 1.2.** Suppose \( k \geq 1, c \geq 0 \) and \( n \geq 2c + k + 1 \).

(i) If \( n \geq 3c + k + 1 \), then \( F_n(k, C_3^{(c)}) \) is the unique extremal graph in \( C(n, k, c) \) of second type.

(ii) If \( n = 2c + k + 1 + t \) and \( 0 \leq t \leq c - 1 \), then \( F_n(k, C_4^{(t)}, C_3^{(c-t)}) \) is the unique extremal graph in \( C(n, k, c) \) of second type.

2. **Some preliminaries.** The graph \( W_G(uv) \) is obtained from \( G \) by subdividing the edge \( uv \), i.e., adding a new vertex \( w \) and edges \( uw, uv \) in \( G - uv \), where \( uv \in E(G) \).

An internal path, say \( P = v_1 \cdots v_{s+1} \) (\( s \geq 1 \)), is a path joining \( v_1 \) and \( v_{s+1} \) (which need not be distinct) such that the degrees of \( v_1 \) and \( v_{s+1} \) are greater than 2, while all other vertices \( v_2, \ldots, v_s \) are of degree 2.

**Lemma 2.1.** [[10]] Let \( uv \) be an edge in an internal path of a connected graph \( G \). Then, \( \mu(G) > \mu(W_G(uv)) \).

Suppose \( v \) is a vertex of \( G \) with at least two vertices. Let \( G_{l,t} \) (\( l \geq t \geq 2 \)) be the graph obtained from \( G \) by attaching two new paths \( P_l = v_1v_2 \cdots v_l \) and \( P_t = u_1u_2 \cdots u_t \), respectively, to \( v \) of \( G \). Let \( G_{t-1,l+1} = G_{t,l} - uv + uv_v \).

**Lemma 2.2.** [[10]] Let \( G \) be a connected graph with at least two vertices. If \( l \geq t \geq 2 \), then \( \mu(G_{l,t}) > \mu(G_{t-1,l+1}) \).

Let \( m(v) \) denote the average of the degrees of the vertices adjacent to \( v \), i.e., \( m(v) = \sum_{uv \in N(v)} d(u)/d(v) \). Next we shall introduce some bounds for \( \lambda(G) \) and \( \mu(G) \), which will play prominent roles in the proof of our main results.

**Lemma 2.3.** [[12] [13]] If \( G \) is a connected graph with \( n \) vertices, then \( \mu(G) \geq \lambda(G) \geq d_1 + 1 \), where the first equality holds if and only if \( G \) is bipartite, and the second equality holds if and only if \( d_1 = n - 1 \).

**Lemma 2.4.** [[6] [13]] If \( G \) is connected, then

\[
\mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)},
\]

where the equality holds if and only if \( G \) is regular or a star or a path with four vertices.

**Lemma 2.5.** [[7] [13]] If \( G \) is connected, then

\[
\mu(G) \leq \max \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)} : uv \in E(G) \right\},
\]

where the equality holds if and only if \( G \) is regular or a star or a path with four vertices.
where the equality holds if and only if $G$ is regular or a bipartite semiregular graph.

**Lemma 2.6.** [2] Let $v$ be a vertex of a connected graph $G$. Suppose that $v_1, \ldots, v_s$ are pendant vertices of $G$ which are adjacent to $v$. Let $G'$ be the graph obtained from $G$ by adding any $b \left(1 \leq b \leq \frac{n(s-1)}{2}\right)$ edges among $v_1, \ldots, v_s$. Then, $\lambda(G) = \lambda(G')$.

**3. The proofs of Theorems 1.1 and 1.2** The following simple necessary condition turns out to be surprisingly useful in the proof of our main results.

**Lemma 3.1.** Suppose $G$ is a graph of $C(n, k, c)$, where $c \geq 2$ and $k \geq 1$. If either $\lambda(G) \geq k + 2c + 1$ or $\mu(G) \geq k + 2c + 1$, then $G$ is obtained by attaching $k$ paths and $c$ cycles, respectively, to a common vertex.

**Proof.** Suppose the degree sequence of $G$ is $(d_1, d_2, \ldots, d_n)$. Since $G \in C(n, k, c)$, we have $2(n + c - 1) = \sum_{i=1}^{n} d_i$.

If $d_1 + d_2 \geq k + 2c + 3$, then $2(n + c - 1) = \sum_{i=1}^{n} d_i \geq k + 2c + 3 + 2(n - 2 - k) + k = 2n + 2c - 1$, a contradiction. By Lemmas 2.3–2.4 and the facts that $c \geq 2$ and $k \geq 1$, if $d_1 + d_2 \leq k + 2c + 1$, then

$$\lambda(G) \leq \mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} \leq 2 + \sqrt{(d_1 + d_2 - 2)^2} = d_1 + d_2 \leq k + 2c + 1,$$

a contradiction.

Thus, $\lambda(G) \geq k + 2c + 1$ or $\mu(G) \geq k + 2c + 1$ implies that $d_1 + d_2 = k + 2c + 2$. Note that $2(n + c - 1) = \sum_{i=1}^{n} d_i$ and $G$ contains exactly $k$ pendant vertices. So, $d_3 = \cdots = d_{n-k} = 2$ and $d_{n-k+1} = d_{n-k+2} = \cdots = d_n = 1$. Since $d_1 = k + 2c + 2 - d_2 \leq k + 2c$, we divide the proof into the following three cases.

**Case 1.** $d_1 \leq k + 2c - 2$.

By Lemmas 2.3–2.4

$$\lambda(G) \leq \mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} = 2 + \sqrt{(k + 2c)d_1} \leq 2 + \sqrt{(k + 2c)(k + 2c - 2)} < k + 2c + 1,$$

a contradiction.

**Case 2.** $d_1 = k + 2c - 1$.

Then, $d_2 = 3$, which implies that $d(w) \in \{k + 2c - 1, 3, 2, 1\}$ holds for any $w \in V(G)$. By $c \geq 2$ and $k \geq 1$, $G$ is neither a regular nor a bipartite semiregular graph. According to Lemmas 2.3 and 2.4, we have

$$\lambda(G) \leq \mu(G) < \max \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)}, wv \in E(G) \right\}, \quad (3.1)$$
Suppose that \( f(u_0v_0) = \max \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)}, \quad uv \in E(G) \right\} \) occurs at the edge \( u_0v_0 \), where \( d(u_0) \geq d(v_0) \). Then, \( d(u_0) \in \{ k + 2c - 1, 3, 2 \} \), as \( G \) is connected and \( c \geq 2 \).

**Subcase 2.1.** \( d(u_0) = k + 2c - 1 \).

Then, \( d(u_0) = d_1 \geq 4 > d(v_0) \).

If \( d(v_0) = 3 \), then \( d(u_0) = d_1 \) and \( d(v_0) = d_2 \). By inequality (3.1),

\[
\mu(G) < f(u_0v_0) \leq \frac{d_1^2 + d_2 + 2(d_1 - 1) + d_2^2 + d_1 + 2(d_2 - 1)}{d_1 + d_2} = k + 2c - 1 + \frac{14}{k + 2 + 2c} \leq k + 2c + 1,
\]

a contradiction.

If \( d(v_0) = 2 \), then by inequality (3.1),

\[
\mu(G) < f(u_0v_0) \leq \frac{d_1^2 + 3 + 2(d_1 - 1) + 4 + d_1 + 3}{d_1 + 2} = k + 2c + \frac{6}{k + 2c + 1} \leq k + 2c + 1,
\]

a contradiction.

If \( d(v_0) = 1 \), then by inequality (3.1),

\[
\mu(G) < f(u_0v_0) \leq \frac{d_1^2 + 1 + 3 + 2(d_1 - 2) + 1 + d_1}{d_1 + 1} = k + 2c + 1 - \frac{1}{k + 2c} < k + 2c + 1,
\]

a contradiction.

**Subcase 2.2.** \( d(u_0) = 3 \).

Since \( d_1 \geq 4 \) and \( d(u_0) = 3 = d_2 > d_3 \), we have \( 1 \leq d(v_0) \leq 2 \).

If \( d(v_0) = 2 \), then by inequality (3.1),

\[
\mu(G) < f(u_0v_0) \leq \frac{d_2^2 + d_1 + 2(d_2 - 1) + 4 + d_1 + d_2}{d_2 + 2} = \frac{2(k + 2c) + 18}{5} < k + 2c + 1,
\]

a contradiction.

If \( d(v_0) = 1 \), then by inequality (3.1),

\[
\mu(G) < f(u_0v_0) \leq \frac{d_2^2 + d_1 + 1 + 1 + d_2}{d_2 + 1} = \frac{k + 2c + 15}{4} < k + 2c + 1,
\]

a contradiction.
Subcase 2.3. $d(u_0) = 2$.

Then, $1 \leq d(v_0) \leq 2$.

If $d(v_0) = 2$, then by inequality (3.1),

$$\lambda(G) \leq \mu(G) < f(u_0v_0) \leq \frac{2(4 + d_1 + 2)}{2 + 2} = \frac{k + 2c + 5}{2} \leq k + 2c + 1,$$

a contradiction.

If $d(v_0) = 1$, then by inequality (3.1),

$$\lambda(G) \leq \mu(G) < f(u_0v_0) \leq \frac{4 + d_1 + 1 + 1 + 2}{2 + 1} = \frac{k + 2c + 7}{3} < k + 2c + 1,$$

a contradiction.

Case 3. $d_1 = k + 2c$.

Then, $d_2 = 2$, and hence $d_2 = \cdots = d_{n-k} = 2$ and $d_{n-k+1} = \cdots = d_n = 1$, which implies that $G$ is obtained by attaching $k$ paths and $c$ cycles to a common vertex. □

Proof of Theorem [11]. When $0 \leq c \leq 1$, the result had been proved in [8] [10] [13] [15]. So, we may suppose that $c \geq 2$ and $G$ is an extremal graph in $C(n, k, c)$ of first type in the sequel.

Since $F_n(k, C_3^{(c)}) \in C(n, k, c)$ and $F_n(k, C_3^{(c)})$ is non-bipartite, by the choice of $G$ and Lemma 2.4, $\mu(G) \geq \mu(F_n(k, C_3^{(c)})) > k + 2c + 1$. Thus, by Lemma 3.1, $G$ is obtained by attaching $k$ paths and $c$ cycles, respectively, to a common vertex, say $u_0$.

Suppose that $G$ contains a cycle, say $C$, of length at least four. Let $u, v$ and $w$ be three vertices of $C$ such that $uv \in E(C)$, $vw \in E(C)$ and $u_0 \notin \{u, v, w\}$. Suppose $x$ is a pendant vertex of $G$. Let $G_1 = G + uw - uv - vw$, $G_2 = G_1 - v$ and let $G_3 = G_1 + xv$. Then, $G_3 \in C(n, k, c)$. Since $c \geq 2$, $uv$ lies on an internal path of $G_2$ and $G = W_{G_2}(uw)$. By Lemma 2.4, $\mu(G) < \mu(G_2)$. Furthermore, since $G_1 \subset G_3$, we have $\mu(G_1) < \mu(G_3)$. Thus, $\mu(G) < \mu(G_2) < \mu(G_1) < \mu(G_3)$, contrary to the choice of $G$. So, every cycle of $G$ is a triangle, and hence $G$ is obtained by attaching $k$ paths (say $P_1, P_2, \ldots, P_k$, where $l_i \geq 2$ for $1 \leq i \leq k$) and $c$ triangles, respectively, to $u_0$.

If there exists two paths, the length of which differ at least two, without loss of generality, we suppose that $l_1 - l_2 \geq 2$. Suppose $P_1 = u_0w_1w_2 \cdots w_{l_1} - 1$ and $P_2 = u_0z_1z_2 \cdots z_{l_2} - 1$. Let $G_5 = G - w_{l_1 - 2}w_{l_1 - 1} + z_{l_2 - 2}z_{l_2 - 1}$. Then, $G_5 \in C(n, k, c)$. By Lemma 2.2, we have $\mu(G) < \mu(G_5)$, which contradicts the choice of $G$. Thus, the $k$ paths have almost equal lengths and hence the result follows. □

Suppose $P_i = u_iu_2 \cdots u_i$ is a path. If $d(u_2) = d(u_3) = \cdots = d(u_{i-1}) = 2$ and $d(u_i) = 1$, then $P_i$ is called a pendant path. Denote by $g(G)$ the girth, i.e., the length
of a shortest cycle of \( G \). Let \( F^*_n(k, C_4^{(c-s)}, C_3^{(s)}) \) be the connected \((c-s)\)-cyclic graph obtained from \( F_n(k, C_4^{(c-s)}, C_3^{(s)}) \) by deleting \( s \) edges, the degrees of whose end vertices are two, in the \( s \) triangles of \( F_n(k, C_4^{(c-s)}, C_3^{(s)}) \). In other words, \( F^*_n(k, C_4^{(c-s)}, C_3^{(s)}) \) is obtained from \( F_{n-2s}(k, C_4^{(c-s)}) \) by attaching \( 2s \) pendant edges to the vertex of degree \( k + 2(c-s) \) of \( F_{n-2s}(k, C_4^{(c-s)}) \).

**Lemma 3.2.** Suppose \( G \) is a connected \( c \)-cyclic graph on \( n \) vertices obtained by attaching \( k \) paths, \( s \) triangles, and \( c-s \) cycles of order at least four, respectively, to a common vertex, where \( 1 \leq s \leq c \) and \( k \geq 1 \). Then,

\[
\lambda(G) \leq \lambda(F^*_n(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})),
\]

where the first equality holds if and only if \( G = F_n(k, C_4^{(c-s)}, C_3^{(s)}) \).

**Proof.** Let \( G_1 \) be the connected \((c-s)\)-cyclic graph obtained from \( G \) by deleting \( s \) edges, the degrees of whose end vertices are two, in the \( s \) triangles of \( G \). Then, \( g(G_1) \geq 4 \). Suppose \( u_0 \) has the maximum degree of \( G \). Then, \( d(u_0) = k + 2c \). By Lemma 2.2

\[
\lambda(F^*_n(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) \text{ and } \lambda(G) = \lambda(G_1).
\]

We consider the following two cases.

**Case 1.** The length of every cycle of \( G_1 \) is four.

Then, \( G_1 \) is a connected \((c-s)\)-cyclic graph obtained by attaching \( k + 2s \) paths (among which at least \( 2s \) paths have lengths 1), and \( c-s \) quadrilaterals, respectively, to \( u_0 \). Suppose that \( P_{l_1}, P_{l_2}, \ldots, P_{l_k} \) are \( k \) pendant paths of \( G_1 \) with the first \( k \) largest lengths among all the pendant paths of \( G_1 \). If there exists two pendant paths of \( \{P_{l_1}, P_{l_2}, \ldots, P_{l_k}\} \), the length of which differ at least two, without loss of generality, we may suppose that \( l_1 - l_2 \geq 2 \). Suppose \( P_{l_1} = u_0w_1w_2 \cdots w_{l_1-1} \) and \( P_{l_2} = u_0z_1z_2 \cdots z_{l_2-1} \). Let \( G_2 = G_1 - w_1l_1-2w_{l_1-1} + z_{l_2-1}w_{l_2-1} \). By Lemma 2.2,

\[
\mu(G_1) < \mu(G_2).
\]

Repeating the above process, we see by Lemma 2.2 that \( \mu(G_1) \leq \mu(F^*_n(k, C_4^{(c-s)}, C_3^{(s)})) \), where the equality holds if and only if \( G_1 = F^*_n(k, C_4^{(c-s)}, C_3^{(s)}) \) namely, \( G = F_n(k, C_4^{(c-s)}, C_3^{(s)}) \).

Since \( \lambda(F^*_n(k, C_4^{(c-s)}, C_3^{(s)})) \) is bipartite, \( \mu(F^*_n(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F^*_n(k, C_4^{(c-s)}, C_3^{(s)})) \) by Lemma 2.4. Thus, by Lemma 2.4 we have

\[
\lambda(G) = \lambda(G_1) \leq \mu(G_1) \leq \mu(F^*_n(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F^*_n(k, C_4^{(c-s)}, C_3^{(s)})).
\]

If \( \lambda(G) = \lambda(F^*_n(k, C_4^{(c-s)}, C_3^{(s)})) \), then \( \mu(G_1) = \mu(F^*_n(k, C_4^{(c-s)}, C_3^{(s)})) \), and hence \( G = F_n(k, C_4^{(c-s)}, C_3^{(s)}) \). Conversely, if \( G = F_n(k, C_4^{(c-s)}, C_3^{(s)}) \), then \( \lambda(G) = \lambda(F^*_n(k, C_4^{(c-s)}, C_3^{(s)})) \).

**Case 2.** \( G_1 \) contains a cycle, say \( C \), of length at least five.

Let \( u, v \) and \( w \) be three vertices of \( C \) such that \( uv \in E(C) \), \( vw \in E(C) \) and \( u_0 \notin \{u, v, w\} \). Let \( P \) be a longest pendant path of \( G_1 \) with initial vertex \( u_0 \), and
let $x$ be the pendant vertex of $P$. Let $G_2 = G_1 + uw - vw - vw$, $G_3 = G_2 - v$ and let $G_4 = G_2 + vex$. Since $d(u_0) = k + 2c + 3$, $uw$ lies on an internal path of $G_3$ and $G_1 = W_{C_{4}}(uw)$. By Lemma 2.4, $\mu(G_1) < \mu(G_3)$. Furthermore, since $G_2 \subseteq G_4$, we have $\mu(G_2) < \mu(G_4)$. Thus, $\mu(G_1) < \mu(G_3) = \mu(G_2) < \mu(G_4)$.

Note that $G_4$ contains exactly $k + 2s$ pendant vertices, and there are at least 2 pendant vertices being adjacent to $u_0$ in $G_4$. By repeating the above process, we see that there exists some $(c - s)$-cyclic graph, say $G_5$, such that $\mu(G_4) \leq \mu(G_5)$, where $G_5$ is obtained by attaching $k + 2s$ paths (among which at least $2s$ paths have lengths 1) and $c - s$ quadrilaterals, respectively, to $u_0$. By the proof of Case 1, we have $\mu(G_5) \leq \mu(F_n(k, C_4^{(c-s)}, C_3^{(s)}))$.

Since $F_n^*(k, C_4^{(c-s)}, C_3^{(s)})$ is bipartite, Lemma 2.3 implies that $\mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$. Now, by Lemma 2.3, we can conclude that

$$\lambda(G) = \lambda(G_1) \leq \mu(G_1) < \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$$

This completes the proof of this result. \[ \square \]

**Lemma 3.3.** Suppose $1 \leq s \leq c$, $k \geq 1$ and $n \geq k + 3c + 2 - s$. Then, $\lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) < \lambda(F_n(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$.

**Proof.** Suppose $u_0$ has the maximum degree of $F_n^*(k, C_4^{(c-s)}, C_3^{(s)})$. Then, $d(u_0) = k + 2c$. Let $P$ be a longest pendant path of $F_n^*(k, C_4^{(c-s)}, C_3^{(s)})$ with initial vertex $u_0$, and let $y$ be the pendant vertex of $P$. If $|V(P)| = 2$, then $n \leq 2s + 3(c - s) + 1 + k = k + 3c + 1 - s$, a contradiction. Thus, $|V(P)| \geq 3$. Let $x$ be a pendant vertex, which is adjacent to $u_0$.

Let $G_1 = F_n^*(k, C_4^{(c-s)}, C_3^{(s)} + xy$. Then, $g(G_1) \geq 4$ and $G_1$ is obtained by attaching $k + 2(s - 1)$ paths (among which at least 2($s - 1$) paths have lengths 1), $c - s$ quadrilaterals and a cycle (say $C_q$, where $q \geq 4$), respectively, to $u_0$.

If $q = 4$, then by Lemma 2.2, $\mu(G_1) \leq \mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$. If $q \geq 5$, then since $d(u_0) = k + 2c + 3$, by Lemma 2.1, there exists some graph, say $G_2$, such that $\mu(G_1) < \mu(G_2)$, where $G_2$ is obtained by attaching $k + 2(s - 1)$ paths (among which at least 2($s - 1$) paths have lengths 1) and $c - s + 1$ quadrilaterals, respectively, to $u_0$. Now, Lemma 2.2 implies that $\mu(G_2) \leq \mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$. Thus, $\mu(G_1) \leq \mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$.

Note that $F_n^*(k, C_4^{(c-s)}, C_3^{(s)}) \subseteq G_1$. Then, $\mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) < \mu(G_1)$. By Lemma 2.6, we have $\lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) < \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$. Also, $\lambda(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})) = \lambda(F_n(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$. Since $F_n^*(k, C_4^{(c-s)}, C_3^{(s)})$ and $F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})$ are two bipartite graphs, by Lemma 2.3, $\lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$ and $\mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})) = \lambda(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$. 


Now, we can conclude that
\[
\lambda(F_n(k, C_{4}^{(c-s)}, C_{3}^{(s)})) = \lambda(F_n^{*}(k, C_{4}^{(c-s)}, C_{3}^{(s)})) = \mu(F_n^{*}(k, C_{4}^{(c-s)}, C_{3}^{(s)})) < \mu(G_1) \\
= \mu(F_n^{*}(k, C_{4}^{(c-s+1)}, C_{3}^{(s-1)})) \\
= \lambda(F_n^{*}(k, C_{4}^{(c-s+1)}, C_{3}^{(s-1)})) = \lambda(F_n(k, C_{4}^{(c-s+1)}, C_{3}^{(s-1)})).
\]
Thus, \( \lambda(F_n(k, C_{4}^{(c-s)}, C_{3}^{(s)})) < \lambda(F_n(k, C_{4}^{(c-s+1)}, C_{3}^{(s-1)})) \). \( \square \)

Proof of Theorem 1.2 (i). When \( 0 \leq c \leq 1 \), the result had been proved in [3][8][11][14]. So, we may suppose that \( c \geq 2 \) and \( G \) is an extremal graph in \( \mathbb{C}(n, k, c) \) of second type in the sequel.

Note that \( F_n(k, C_{4}^{(c)}) \in \mathbb{C}(n, k, c) \). By Lemma 2.4 and the choice of \( G, \lambda(G) \geq \lambda(F_n(k, C_{4}^{(c)})) > k + 2c + 1 \). Thus, \( G \) is obtained by attaching \( k \) paths and \( c \) cycles, respectively, to a common vertex by Lemma 3.1. We consider the following two cases.

Case 1. \( g(G) \geq 4 \).

If every cycle of \( G \) is a quadrilateral, by Lemma 2.2 it follows that \( \mu(G) \leq \mu(F_n(k, C_{4}^{(c)})) \), where the equality holds if and only if \( G = F_n(k, C_{4}^{(c)}) \). If \( G \) contains at least one cycle of length at least five, since \( c \geq 2 \), by Lemma 2.1 there exists some \( c \)-cyclic graph, say \( G_1 \), such that \( \mu(G) < \mu(G_1) \), where \( G_1 \) is obtained by attaching \( k \) paths and \( c \) quadrilaterals, respectively, to a common vertex. Furthermore, Lemma 2.2 implies that \( \mu(G_1) \leq \mu(F_n(k, C_{4}^{(c)})) \), and hence, \( \mu(G) < \mu(F_n(k, C_{4}^{(c)})) \).

So, we can conclude that \( \mu(G) \leq \mu(F_n(k, C_{4}^{(c)})) \), where the equality holds if and only if \( G = F_n(k, C_{4}^{(c)}) \). Since \( F_n(k, C_{4}^{(c)}) \) is bipartite, by Lemma 2.3 we have \( \lambda(G) \leq \mu(G) \leq \mu(F_n(k, C_{4}^{(c)})) = \lambda(F_n(k, C_{4}^{(c)})) \). Now, if \( \lambda(G) = \lambda(F_n(k, C_{4}^{(c)})) \), then \( \mu(G) = \mu(F_n(k, C_{4}^{(c)})) \), and hence, \( G = F_n(k, C_{4}^{(c)}) \).

Case 2. \( g(G) = 3 \).

We may suppose that \( G \) contains exactly \( s \geq 1 \) triangles. By Lemma 3.2, \( \lambda(G) \leq \lambda(F_n(k, C_{4}^{(c-s)}, C_{3}^{(s)})) \). Since \( s \geq 1 \), Lemma 3.3 implies that
\[
\lambda(F_n(k, C_{4}^{(c-s)}, C_{3}^{(s)})) < \lambda(F_n(k, C_{4}^{(c-s+1)}, C_{3}^{(s-1)})) \leq \cdots \leq \lambda(F_n(k, C_{4}^{(c)}, C_{3}^{(0)})).
\]
Thus, \( \lambda(G) < \lambda(F_n(k, C_{4}^{(c)}, C_{3}^{(0)})) = \lambda(F_n(k, C_{4}^{(c)})) \). \( \square \)

Proof of Theorem 1.2 (ii). When \( c = 1 \), then \( n = k + 3 \), and the result clearly follows. So, we may suppose that \( c \geq 2 \) and \( G \) is an extremal graph in \( \mathbb{C}(n, k, c) \) of second type in the sequel. Since \( F_n(k, C_{4}^{(t)}, C_{3}^{(c-t)}) \in \mathbb{C}(n, k, c) \), by Lemma 2.3 and the choice of \( G, \lambda(G) \geq \lambda(F_n(k, C_{4}^{(t)}, C_{3}^{(c-t)})) \geq k + 2c + 1 \). Thus, \( G \) is obtained by attaching \( k \) paths and \( c \) cycles, respectively, to a common vertex by Lemma 3.1.

Note that \( n \leq 3c + k \). So, we may suppose that \( G \) contains exactly \( s \geq 1 \) triangles.
If \( s \leq c - t - 1 \), then
\[
\begin{align*}
n &\geq 2s + 3(c - s) + k + 1 = 3c + k + 1 - s \\
&\geq 3c + k + 1 - (c - t - 1) = k + 2c + t + 2,
\end{align*}
\]
a contradiction. Thus, \( s \geq c - t \).

If \( s = c - t \), then by Lemma 3.2, \( \lambda(G) \leq \lambda(F_n(k, C_4^{(c-t)}, C_3^{(c-t)})) \), where the equality holds if and only if \( G = F_n(k, C_4^{(c-t)}, C_3^{(c-t)}) \). If \( s \geq c - t + 1 \), then since \( n = 2c + k + 1 + t \geq 3c + k + 2 - s \), by Lemmas 3.2–3.3 we can conclude that
\[
\lambda(G) \leq \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) < \lambda(F_n(k, C_4^{(c-s+1)}, C_3^{(s-1)})) \leq \cdots \leq \lambda(F_n(k, C_4^{(c-t)}, C_3^{(c-t)})).
\]

4. Further discussion. By Theorems 1.1 and 1.2, the unique extremal graph in \( C(n, k, c) \) of first type and the unique extremal graph in \( C(n, k, c) \) of second type are, respectively, determined for \( c \geq 0 \), \( k \geq 1 \) and \( n \geq 2c + k + 1 \).

When \( c \geq 0 \), \( k \geq 1 \) and \( n \leq 2c + k \), for any \( G \in C(n, k, c) \), by Lemma 2.3 we have \( \lambda(G) \leq n \), where the equality holds if and only if \( d_1 = n - 1 \). Furthermore, when \( c \geq 3 \), the extremal graphs in \( C(n, k, c) \) of second type are always not unique. For instance, let \( W_1 \) and \( W_2 \) be the two tricyclic graphs on \( n \) vertices as shown in Fig. 4.1. Then, \( \{W_1, W_2\} \subseteq C(n, k, 3) \). Since \( d_1(W_1) = n - 1 = d_1(W_2) \), by Lemma 2.3 we have \( \lambda(W_1) = n = \lambda(W_2) \).

When \( c \geq 0 \), \( k \geq 1 \) and \( n \leq 2c + k \), it is still an open problem to characterize the extremal graphs in \( C(n, k, c) \) of first type.

![Fig. 4.1. The tricyclic graphs \( W_1 \) and \( W_2 \).](image)

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REFERENCES