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Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1628

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TPₖ COMPLETION OF PARTIAL MATRICES
WITH ONE UNSPECIFIED ENTRY

V. AKIN¹, C.R. JOHNSON², AND S. NASSERASR³

Abstract. Every partial TP₂ (TP₁) matrix with one unspecified entry has a TP₂ (TP₁) completion. For a given m-by-n pattern with one unspecified entry, the minimum set of conditions characterizing TP₃ completability is given. These conditions are at most eight polynomial inequalities on the specified entries of the pattern. For k ≥ 3, patterns with one unspecified entry that are TPₖ completable are also characterized, and conditions are described for completability otherwise.

Key words. Totally positive-k matrix, Completion problem, Pattern, Polynomial conditions.

AMS subject classifications. 15A83, 15B48.

1. Introduction. An m-by-n matrix A is said to be TPₖ (totally positive-k) if all its ℓ-by-ℓ minors are positive, for ℓ = 1, . . . , k. If k = min{m, n}, so that all minors are positive, then A is called totally positive. A partial matrix is one in which some entries are specified, while the remaining, unspecified, entries are free to be chosen. A partial TPₖ matrix is a partial matrix all of whose fully specified minors of order at most k are positive. A completion of a partial matrix is a particular choice of values for the unspecified entries, resulting in a conventional matrix. The pattern (which we may think of as an array) of a partial matrix is an inventory of which entries are specified (and which are unspecified). A pattern P of specified entries is called TPₖ completable if every partial TPₖ matrix with pattern P has a TPₖ completion. A matrix completion problem asks which partial matrices have a completion enjoying an identified property. In studying the completion problem for TP matrices, understanding the completion problem for TPₖ matrices for k = 1, . . . , n has proven to be helpful. It is obvious that a partial TP₁ matrix is always TP₁ completable. The problem of TP₂ completion has been completely solved in [6]. In particular, it is known that any pattern with only one unspecified entry is TP₂ completable. As a next step in this process, we consider the TPₖ completion of patterns with one unspecified entry. It turns out that depending on the location of

¹Received by the editors on November 22, 2012. Accepted for publication on April 30, 2014. Handling Editor: Leslie Hogben.
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the unspecified entry and the value of \( k \), the conditions for \( \text{TP}_k \) completability vary. Our objective is to describe such conditions.

In Section 2 some preliminary results are presented. Section 3 provides a family of pairs of minors that do not produce extra conditions for \( \text{TP}_k \) completability. In Section 4, an explicit description of the minimum set of conditions for \( \text{TP}_3 \) completability of patterns with one unspecified entry is given. In Section 5, a combinatorial characterization of \( \text{TP}_k \) completable patterns, \( k \geq 4 \), with one unspecified entry is given. Finally, in the Appendix, a list of conditions for \( \text{TP}_4 \) completability of a pattern is given.

2. Background. Using Tarski-Seidenberg principle [1] and the fact that the set of \( m \)-by-\( n \) \( \text{TP}_k \) matrices form a semialgebraic set, we know that, for any pattern of specified entries, there is a finite number of polynomial inequalities on the specified entries that characterize the \( \text{TP}_k \) completability of the pattern. However, for a given pattern, neither the polynomial inequalities nor the number of these can be obtained from Tarski-Seidenberg principle. Here, our purpose is to better understand efficient lists of such inequalities for \( \text{TP}_k \) completion of patterns with just one unspecified entry.

For \( \gamma \subseteq \{1, \ldots, n\} \), the notation \( \gamma^c \) is used to denote the set \( \{1, \ldots, n\}\setminus \gamma \). The set of real \( m \)-by-\( n \) matrices is denoted by \( M_{m,n} \), when \( m = n \), we use \( M_n \) instead of \( M_{m,n} \). For \( \alpha \subseteq \{1, \ldots, m\} \) and \( \beta \subseteq \{1, \ldots, n\} \), the submatrix of \( A \) lying in the rows indexed by \( \alpha \) (\( \alpha^c \)) and the columns indexed by \( \beta \) (\( \beta^c \)) is denoted by \( A[\alpha, \beta] \). If \( \alpha \) and \( \beta \) are sets of consecutive numbers, then \( A[\alpha, \beta] \) is called a contiguous submatrix of \( A \). A contiguous minor is defined similarly. A matrix \( A \) is called \( \text{TP}_k \)-contiguous if every contiguous minor of \( A \) of order \( \ell = 1, \ldots, k \) is positive.

The following known result is a useful tool throughout this work, see [2, 3] for a proof.

**Lemma 2.1.** An \( m \)-by-\( n \) matrix \( A \) is \( \text{TP}_k \) if and only if it is \( \text{TP}_k \)-contiguous.

Suppose \( A \) is an \( m \)-by-\( n \) partial \( \text{TP}_k \) matrix with one unspecified entry \( x \) in the position \((i, j)\). Let \( C_x \) be the set of all contiguous square submatrices of \( A \) of order \( 1, \ldots, k \) that contain the unspecified entry \( x \). The submatrix \( S_{ij} \) of \( A \) that contains all of the submatrices in \( C_x \) and has the minimum order is called the \( k \)th surrounding submatrix of \( A \) with respect to \((i, j)\). Thus, the order of \( S_{ij} \) is at most \((2k - 1)\)-by-(\(2k - 1\)). Using Lemma 2.1 we have the following result that we use frequently.

**Lemma 2.2.** Let \( A \) be an \( m \)-by-\( n \) partial \( \text{TP}_k \) matrix with one unspecified entry \( x \) in the position \((i, j)\). Then \( A \) is \( \text{TP}_k \) completable if and only if \( S_{ij} \) is \( \text{TP}_k \) completable.
Note that, $S_{ij}$ is TP$_k$ completable if and only if there is a value $a_{ij}$ such that replacing $x$ by $a_{ij}$, implies positivity of every minor in $C_x$. Consider $B = [b_{uv}] \in C_x$, and suppose $x$ is in the $(r, s)$ position of $B$. Then, by expanding the determinant of $B$ along its $r$th row, we have

\begin{equation}
(2.1) \quad \det(B) > 0 \iff (-1)^{r+s}x \det(B_{rs}) + \sum_{p=1\atop p \neq s}^{\ell} (-1)^{r+p}b_{rp} \det(B_{rp}) > 0
\end{equation}

in which $B_{rp}$ is the submatrix of $B$ obtained by deleting the $r$th row and $p$th column. Since $B$ is partial TP$_k$, $\det(B_{rs}) > 0$. Therefore, if $r + s$ is odd, then (2.1) gives an upper bound for $x$ and if $r + s$ is even, then (2.1) gives a lower bound for $x$. Let $L_{ij}$ be the set of all lower bounds for $x$ obtained from the inequalities in (2.1) and $U_{ij}$ be the set of all upper bounds for $x$ obtained from the inequalities in (2.1), for each $B \in C_x$. In order to have a TP$_k$ completion, for every $\ell \in L_{ij}$ and every $u \in U_{ij}$, the inequality $\ell < u$ should hold. This implies the following result.

**Theorem 2.3.** Consider the partial TP$_k$ matrix $A$ with one unspecified entry in the position $(i, j)$. Then, $A$ is TP$_k$ completable if and only if

$$\max\{\ell\}_{\ell \in L_{ij}} < \min\{u\}_{u \in U_{ij}}.$$

We note that because TP$_k$ contiguous is sufficient for TP$_k$, Lemma 2.2 and Theorem 2.3 can be applied to patterns with more than one unspecified entry as long as they are far enough apart. For instance, suppose the pattern $P$ has only two unspecified entries $(p, q)$ and $(r, s)$, where $S_{pq}$ and $S_{rs}$ do not share any entry ($S_{pq} \cap S_{rs} = \emptyset$), then using Lemma 2.1, $P$ is TP$_k$ completable if and only if each of $S_{pq}$ and $S_{rs}$ is TP$_k$ completable. This can be generalized as the following.

**Lemma 2.4.** Suppose $P$ is an $m$-by-$n$ pattern with $t$ unspecified entries in the positions $(i_u, j_u)$, $u = 1, \ldots, t$. If for all $u, v \in \{1, \ldots, t\}$, with $u \neq v$, we have $S_{i_u, j_u} \cap S_{i_v, j_v} = \emptyset$, then $P$ is TP$_k$ completable if and only if each $S_{i_u, j_u}, u \in \{1, \ldots, t\}$ is TP$_k$ completable.

Some of the inequalities in Theorem 2.3 are redundant, a family of such inequalities is described in the next section.

### 3. Unconditional pairs of minors.

In the previous section, we presented the general conditions for TP$_k$ completability of patterns, when the unspecified entries are far apart. However, computing all these conditions can be a tedious work. Providentially, in a partial TP$_k$ matrix some of these lower bounds are always less than some of the upper bounds. A family of such pairs of bounds are described in this section.
Consider a pattern $P$ with exactly one unspecified entry $x$ in the position $(i,j)$. Let $P_1$ and $P_2$ be two subpatterns of $P$ containing $x$ where the sum of the indices of $x$ in $P_1$ is even and the sum of the indices of $x$ in $P_2$ is odd. Suppose $A$ is a partial TP$_k$ matrix with pattern $P$. Then, for the submatrix $A_1$ corresponding to $P_1$, the inequality $\det A_1 > 0$ implies a lower bound for $x$, say $L_{A_1}$, and for the submatrix $A_2$ corresponding to $P_2$, the inequality $\det A_2 > 0$ implies an upper bound for $x$, say $U_{A_2}$. If for any partial TP$_k$ matrix $A$ with pattern $P$, $L_{A_1} < U_{A_2}$, then the pair of subpatterns $P_1$ and $P_2$ is called an unconditional pair of subpatterns. Otherwise, such a pair is called a conditional pair of subpatterns. We use the term conditional (unconditional) minors (or submatrices) when it refers to minors (or submatrices). A trivial example for an unconditional pair of minors is a pair of 2-by-2 minors, since any partial TP$_2$ pattern with one unspecified entry is TP$_2$ completable. In this section, for a given pattern with one unspecified entry, a family of pairs of unconditional subpatterns is determined.

The following lemma is proved in [3].

**Lemma 3.1.** Let $\hat{T}_\ell = [t_{ij}] \in M_\ell$ be a permutation matrix with $t_{i,\ell-i+1} = 1$, for $i = 1, \ldots, \ell$. If $A \in M_{m,n}$ is TP$_k$, then both $A^T$ and $\hat{T}_m A \hat{T}_n$ are TP$_k$.

If a pair of minors is conditional (unconditional), then depending on their situation and using Lemma 3.1 there can be three more pairs of conditional (unconditional) minors. This is useful in this work.

The following lemma can be obtained by the short-term Plücker identity; see [3]. We present a different proof using Sylvester’s identity.

**Lemma 3.2.** Let $A \in M_{n,n+2}$, and $j, k \in \{2, \ldots, n+1\}$. Suppose $\alpha = \{1, \ldots, n\}$, $\beta = \{1, \ldots, n+2\} \setminus \{j, k\}$. If $j < k$, then
\[
\det A[\alpha, \{1, k\}] \det A[\alpha, \{j, n+2\}] - \det A[\alpha, \{1, j\}] \det A[\alpha, \{k, n+2\}] = \det A[\alpha, \{j, k\}] \det A[\alpha, \{1, n+2\}].
\]

**Proof.** Let $r_j$ and $r_k$ be row vectors in $\mathbb{R}^{n+2}$ where $r_j[1] = (-1)^{1+j}$, $r_k[n+2] = (-1)^{n+k}$, and all other entries of both vectors are zero. Insert row $r_j$ between rows $j-1$ and $j$ of $A$ and $r_k$ between rows $k-2$ and $k-1$ of $A$ to obtain the matrix $A'$. Rename the rows of $A'$ from $1, \ldots, j-1, r_j, j, j+1, \ldots, k-2, r_k, k-1, \ldots, n$ to $1, 2, \ldots, n+2$, with the same order. Then, $A'$ is of the following form, where $A'[\beta, \{1, \ldots, n+2\}] = A$. 


Now, the matrix $B \in M_2$ with 

$$
B = \begin{bmatrix}
\det A'[^{\beta \cup \{j\}, \beta \cup \{j\}}] & \det A'[^{\beta \cup \{j\}, \beta \cup \{k\}}] \\
\det A'[^{\beta \cup \{k\}, \beta \cup \{j\}}] & \det A'[^{\beta \cup \{k\}, \beta \cup \{k\}}]
\end{bmatrix}
$$

satisfies the Sylvester's identity (see [5]):

$$
\det B = (\det A[^{\beta \cup \{j\}}])^{(n+2)-n-1} \det A' = \det A[^{\alpha, \{j, k\}^c}] \det A[^{\alpha, \{1, n + 2\}^c}].
$$

So, 
\[
\det A[^{\alpha, \{1, k\}^c}] \det A[^{\alpha, \{j, n + 2\}^c}] - \det A[^{\alpha, \{1, j\}^c}] \det A[^{\alpha, \{k, n + 2\}^c}]
\]

\[
= \det A[^{\alpha, \{j, k\}^c}] \det A[^{\alpha, \{1, n + 2\}^c}].
\]

Using the above lemma, a family of unconditional pairs of minors is given as follows.

**Lemma 3.3.** Consider a partial TP$_k$ matrix $A$ with only one unspecified entry $x$ in the position $(i, j)$. Suppose two $\ell$-by-$\ell$, $\ell \leq k$, contiguous submatrices of $A$ lie in rows $r_1, r_1 + 1, \ldots, r_1 + \ell - 1$ and columns $c_1, c_1 + 1, \ldots, c_1 + \ell$. If $j \in \{c_1 + 1, \ldots, c_1 + \ell - 1\}$, then the lower bound obtained from one of the minors is less than the upper bound obtained from the other minor.

**Proof.** Without loss of generality suppose $A$ is an $\ell$-by-($\ell + 1$) matrix, $\{r_1, r_1 + 1, \ldots, r_1 + \ell - 1\} = \{1, 2, \ldots, \ell\}$, $\{c_1, c_1 + 1, \ldots, c_1 + \ell\} = \{1, 2, \ldots, \ell + 1\}$, and let $i + j$ be even. The proof for $i + j$ odd is similar. By expanding the determinant of
each of the submatrices along the \( i \)th row, we have

\[
x \det A([i], [j, \ell + 1]) + \sum_{\ell+1 \leq t \leq \ell} (-1)^{t+\ell} a_{i,t} \det A([i], [t, \ell + 1]) > 0, \quad \text{and}
-x \det A([i], [1, j]) + \sum_{1 \leq t \leq j} (-1)^{t-\ell} a_{i,t} \det A([i], [1, t]) > 0.
\]

Therefore, there exists \( x \) such that

\[
\frac{\sum_{\ell+1 \leq t \leq \ell} (-1)^{t+\ell-1} a_{i,t} \det A([i], [1, t])}{\det A([i], [1, j])} > x \frac{\sum_{1 \leq t \leq j} (-1)^{t-\ell} a_{i,t} \det A([i], [1, t])}{\det A([i], [1, \ell + 1])}
\]

if and only if

\[
\sum_{\ell+1 \leq t \leq \ell} (-1)^{t+\ell-1} a_{i,t} \det A([i], [1, t]) + (-1)^{\ell+1} a_{i,\ell+1} \det A([i], [1, j]) \det A([i], [1, \ell + 1]) > 0
\]

Using Lemma 3.2, this is equivalent to

\[
\det A([i], [1, \ell + 1]) \bigg[ \sum_{\ell+1 \leq t \leq \ell} (-1)^{t+\ell-1} a_{i,t} \det A([i], [t, j]) \bigg] + (-1)^{\ell+1} a_{i,\ell+1} \det A([i], [1, j]) \det A([i], [1, \ell + 1]) > 0
\]

\[
\Leftrightarrow \sum_{\ell+1 \leq t \leq \ell} (-1)^{t+\ell-1} a_{i,t} \det A([i], [t, j]) + (-1)^{\ell+1} a_{i,\ell+1} \det A([i], [1, j]) \det A([i], [1, \ell + 1]) > 0
\]

The last inequality holds since the matrix is partial TP\(_k\). \( \square \)

Therefore, in studying the TP\(_k\) completability of a given pattern, comparison of lower and upper bounds coming from subpatterns satisfying the assumption of Lemma
need not be made. It is not known whether the remaining pairs of subpatterns are all conditional pairs. However, in the next section, we show that the inequalities obtained from some of the remaining pairs may be implied by the inequalities obtained from others (see the proof of Lemma 4.3 for example). Therefore, the set of inequalities obtained from all of the conditional pairs of subpatterns may not be the minimal set of polynomial inequalities on the specified entries (in addition to being partial TPk) for a pattern to be TPk completable. We call two conditional pairs of subpatterns independent if the conditions obtained from one pair cannot be implied from those of the other pair. The minimum set of independent pairs of conditional subpatterns, for k = 3, is described next.

4. TP3 completion. In this section, necessary and sufficient conditions for TP3 completability of a pattern with one unspecified entry, in the form of minimal polynomial inequalities in the specified entries, are given.

In studying the TP3 completion problem, the smallest order to consider is a pattern of order 3-by-n, n ≥ 3. Note that, a 3-by-n TP3 matrix is also TP, so using Theorem 2.8 of [4], we have the following lemma. We also prove it directly by checking all possible conditional minors.

**Lemma 4.1.** Every partial TP3 matrix of order 3-by-n, n ≥ 3, with exactly one unspecified entry is TP3 completable.

**Proof.** Using Lemma 2.2, it is enough to consider a 3-by-5 partial TP3 matrix A where the unspecified entry lies in the (2,3) position. Using Lemma 3.3, the pairs of minors of order 3 are unconditional. Since A is TP2 completable, the only minors that may produce conditions are $A_{\{1,2,3\}} \setminus \{1,2,3\}$ and $A_{\{2,3\}} \setminus \{2,3\}$. The upper bound/ lower bound condition formed by these minors is

$$\frac{a_{34}a_{33}}{a_{34}} < \frac{a_{13}(a_{21}a_{32} - a_{22}a_{31}) + a_{33}(a_{11}a_{22} - a_{12}a_{21})}{a_{11}a_{32} - a_{12}a_{31}}.$$  \hspace{1cm} (4.1)

Since $a_{33} \det A[\{1,2,3\}, \{1,2,4\}] > 0$, we have

$$a_{33}[a_{14}(a_{21}a_{32} - a_{22}a_{31}) + a_{34}(a_{11}a_{22} - a_{12}a_{21}) - a_{24}(a_{11}a_{32} - a_{12}a_{31})] > 0. \hspace{1cm} (4.2)$$

On the other hand, $a_{13}a_{34} > a_{14}a_{33}$ and $a_{21}a_{32} - a_{22}a_{31} > 0$, thus, using the inequality 4.2, we have

$$a_{13}a_{34}(a_{21}a_{32} - a_{22}a_{31}) + a_{34}a_{33}(a_{11}a_{22} - a_{12}a_{21}) - a_{24}a_{33}(a_{11}a_{32} - a_{12}a_{31}) > 0$$

which is exactly the inequality 4.1. \hfill \Box

Let A be an m-by-n matrix, and suppose $\alpha \subseteq \{1, \ldots, m\}$ and $\beta \subseteq \{1, \ldots, n\}$ with $|\alpha| = |\beta|$. If x is an unspecified entry in the position $(i,j) \in \alpha \times \beta$, then the
The determinant of the submatrix $A[\alpha, \beta]$ is a function of $x$. The value of this function at $x = 0$ is denoted by $\det A[\alpha, \beta](0)$.

**Lemma 4.2.** Let $P$ be a 4-by-4 partial $TP_3$ pattern with exactly one unspecified entry $x$ lying in the position $(i, j)$. Then $P$ is $TP_3$ completable if and only if $(i, j) \neq (2, 3)$, $(3, 2)$. Moreover, for a partial $TP_3$ matrix $A$ with pattern $P$, if $(i, j) = (2, 3)$, then $A$ has a $TP_3$ completion if and only if

$$\frac{-\det A[[2, 3, 4], [1, 2, 3]](0)}{\det A[[3, 4], [1, 2]]} < \frac{\det A[[1, 2], [3, 4]](0)}{\det A[[1], [4]]},$$

and if $(i, j) = (3, 2)$, then $A$ has a $TP_3$ completion if and only if

$$\frac{-\det A[[1, 2, 3], [2, 3, 4]](0)}{\det A[[1, 2], [3, 4]]} < \frac{\det A[[3, 4], [1, 2]](0)}{\det A[[4], [1]]}.$$

**Proof.** If $x$ lies in any of the first or last columns (or rows), then using Lemma 2.2 and Lemma 4.1, the pattern is $TP_3$ completable. If $x$ lies in the $(2, 2)$ position, the only pairs of submatrices that contain $x$ and do not lie in a 3-by-$k$ submatrix, $k = 3, 4$, are the pairs of submatrices $A[[1, 2], [1, 2]], A[[2, 3, 4], [2, 3, 4]]$ and $A[[1, 2, 3], [1, 2, 3]], A[[2, 3, 4], [2, 3, 4]]$. Since the sums of the indices of $x$ have the same parity in both submatrices in each pair, both of these pairs are unconditional. The case $(i, j) = (3, 3)$ is similar.

If $x$ lies in the position $(2, 3)$, then the only pair of submatrices that contains $x$ but do not lie in a 3-by-$k$ submatrix, $k = 3, 4$, is $A[[2, 3, 4], [1, 2, 3]], A[[1, 2], [3, 4]]$. Therefore, the following inequality is a sufficient condition for a partial $TP_3$ matrix $A$ to have a $TP_3$ completion

$$\frac{-\det A[[2, 3, 4], [1, 2, 3]](0)}{\det A[[3, 4], [1, 2]]} < \frac{\det A[[1, 2], [3, 4]](0)}{\det A[[1], [4]]}. \tag{4.3}$$

The following example shows that the above inequality is also a necessary condition for a $TP_3$ completion. Consider the partial $TP_3$ matrix

$$A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 4.9 & x & 8 \\
1 & 6 & 10.9 & 16 \\
1 & 8 & 15 & 31
\end{pmatrix},$$

where the inequality (4.3) does not hold. The inequality $\det A[[2, 3, 4], [1, 2, 3]] > 0$ implies $x > 8.645$, while $\det A[[1, 2], [3, 4]] > 0$ implies $x < 8$. Thus, there is no value for $x$ that forms a $TP_3$ completion for $A$. The proof for the case of $(i, j) = (3, 2)$ is similar. \qed
The following lemma explicitly describes the conditions for \( TP_3 \) completablety of a 5-by-5 partial \( TP_3 \) matrix with one unspecified entry. If the sums of the indices of \( x \) in a submatrix \( B \) is odd (even), then the positivity of \( \det B \) will result in an upper bound (lower bound) for \( x \), we use the notation \( \downarrow (\uparrow) B \) to emphasize this.

**Theorem 4.3.** Let \( A \) be a 5-by-5 partial \( TP_3 \) matrix with one unspecified entry in the position \( (i, j) \). Then

i. if \( i \in \{1, 5\} \) or \( j \in \{1, 5\} \) or \( (i, j) = (2, 2) \) or \( (i, j) = (4, 4) \), then \( A \) has a \( TP_3 \) completion without any extra condition,

ii. if \( (i, j) = (2, 4) \), then \( A \) has a \( TP_3 \) completion if and only if

\[
\frac{-\det A[[2, 3, 4], \{2, 3, 4\]](0)}{\det A[[3, 4], \{2, 3\}]} < \frac{\det A[[1, 2], \{4, 5\]](0)}{\det A[[1], \{5\}]},
\]

if \( (i, j) = (4, 2) \), then \( A \) has a \( TP_3 \) completion if and only if

\[
\frac{-\det A[[2, 3, 4], \{2, 3, 4\]](0)}{\det A[[2, 3], \{3, 4\}]} < \frac{\det A[[4, 5], \{1, 2\]](0)}{\det A[[5], \{1\}]},
\]

iii. if \( (i, j) = (2, 3) \), then \( A \) has a \( TP_3 \) completion if and only if

\[
\frac{-\det A[[2, 3, 4], \{3, 4, 5\]](0)}{\det A[[3, 4], \{4, 5\}]} < \frac{\det A[[1, 2, 3], \{1, 2, 3\]](0)}{\det A[[1, 3], \{1, 2\}]},
\]

and

\[
\frac{-\det A[[2, 3, 4], \{1, 2, 3\]](0)}{\det A[[3, 4], \{1, 2\}]} < \frac{\det A[[1, 2, 3], \{3, 4, 5\]](0)}{\det A[[1, 3], \{4, 5\}]},
\]

the cases \( (i, j) = (3, 2), (3, 4), \) or \( (4, 3) \) are similar.

iv. if \( (i, j) = (3, 3) \), then there are eight polynomial inequalities on the specified entries of \( A \) that need to hold in order for \( A \) to have a \( TP_3 \) completion.

**Proof.**

i. If \( i \in \{1, 5\} \) or \( j \in \{1, 5\} \), using Lemma 2.2, it is enough to consider a 3-by-5 (or 5-by-3) contiguous submatrix containing \( x \), using Lemma 4.1, \( A \) has a \( TP_3 \) completion. If \( (i, j) = (2, 2) \) or \( (4, 4) \), it is enough to consider a 4-by-4 contiguous submatrix \( B \) of \( A \) such that \( x \) lies in the \( (2, 2) \) or \( (3, 3) \) position of \( B \), respectively, using Lemma 4.2, \( A \) has a \( TP_3 \) completion.

ii. If \( (i, j) = (2, 4) \) or \( (4, 2) \), again it is enough to consider a 4-by-4 contiguous submatrix \( B \) of \( A \) containing \( x \) such that \( x \) lies in the \( (2, 3) \) or \( (3, 2) \) position of \( B \), respectively. By Lemma 4.2, conditions for \( TP_3 \) completablety of \( A \) are inequalities given in the statement.

iii. Suppose \( x \) is in the \( (2, 3) \) position. Using Lemma 2.2, the conditional pairs of submatrices lie in \( A[[1, 2, 3, 4], \{1, 2, 3, 4, 5\}] \). The only possible conditional cases
that we need to consider are (a) both submatrices are of order 3; (b) one of the submatrices is of order 2 and the other is 3.

(a) In this case, there are nine such pairs that are listed below:

\[
\begin{align*}
\downarrow A[[1,2,3], \{1,2,3\}], & \uparrow A[[1,2,3], \{2,3,4\}], \\
\downarrow A[[1,2,3], \{1,2,3\}], & \uparrow A[[2,3,4], \{1,2,3\}], \\
\downarrow A[[1,2,3], \{1,2,3\}], & \uparrow A[[2,3,4], \{3,4,5\}], \\
\downarrow A[[1,2,3], \{3,4,5\}], & \uparrow A[[1,2,3], \{2,3,4\}], \\
\downarrow A[[1,2,3], \{3,4,5\}], & \uparrow A[[2,3,4], \{1,2,3\}], \\
\downarrow A[[2,3,4], \{1,2,3\}], & \uparrow A[[2,3,4], \{3,4,5\}], \\
\downarrow A[[2,3,4], \{2,3,4\}], & \uparrow A[[1,2,3], \{2,3,4\}], \\
\downarrow A[[2,3,4], \{2,3,4\}], & \uparrow A[[2,3,4], \{1,2,3\}], \\
\downarrow A[[2,3,4], \{2,3,4\}], & \uparrow A[[2,3,4], \{3,4,5\}].
\end{align*}
\]

(4.4) (4.5)

Except the pairs of submatrices in (4.4) and (4.5), all other seven pairs lie in a 3-by-5 or 5-by-3 submatrix, so using Lemma 4.1 they are unconditional. Moreover, the matrices \(A_1\) and \(A_2\) below show that the two pairs in (4.4) and (4.5) are both conditional and independent. Therefore, the inequalities given in the statement must hold for a TP3 completion.

\[
A_1 = \begin{pmatrix} 10 & 60 & 6 & 5 & 2 \\ 1 & 8.99 & x & 8.61 & 6 \\ 1 & 9 & 8 & 10 & 8.35 \\ 1 & 10 & 11 & 300 & 500 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 2.95 & 5 & 7 \\ 1 & 3.2 & x & 16 & 26.25 \\ 1 & 4 & 12 & 31 & 54 \\ 1 & 5 & 18 & 51 & 97 \end{pmatrix}.
\]

In the partial TP3 matrix \(A_1\), the inequalities \(\det A_1[[1,2,3], \{1,2,3\}] > 0\) and \(\det A_1[[2,3,4], \{3,4,5\}] > 0\) imply \(7.9796 < x < 7.9753\), and in the partial TP3 matrix \(A_2\), the inequalities \(\det A_2[[1,2,3], \{3,4,5\}] > 0\), and \(\det A_2[[2,3,4], \{1,2,3\}] > 0\) imply \(7.2 < x < 7.15542\). Therefore, there is no TP3 completion for any of the matrices \(A_1\) and \(A_2\).

(b) In this case, the only pair with different parities for the sum of indices of \(x\) that are not contained in a 3-by-4 or 3-by-5 submatrix (or the transpose of those) is \(\uparrow A[[2,3,4], \{1,2,3\}], \downarrow A[[1,2], \{3,4\}]\). This can be considered as an unspecified entry in the (2,3) position of a 4-by-4 partial TP3 matrix, using Lemma 4.2 the extra condition for TP3 completion is

\[
(4.6) \quad \frac{-\det A[[2,3,4], \{1,2,3\}]0}{\det A[[3,4], \{1,2\}]} < \frac{\det A[[1,2], \{3,4\}]0}{\det A[[1], \{4\}]} = \frac{a_{13}a_{24}}{a_{14}}.
\]

However, (4.6) is obtained by the inequality produced by matrices in (4.5) and the assumption of partial TP3, as shown below, so it is not an independent condition for having a TP3 completion.
iv. For the case \((i, j) = (3, 3)\), we need to consider the contiguous 5-by-5 matrix. In a similar way to the previous case, we consider two cases:

(a) Suppose both pairs of submatrices are of order 3. There are nine 3-by-3 contiguous submatrices containing \(x\):

\[
\begin{align*}
\text{(4.8)} & \quad \uparrow A[[1, 2, 3], [1, 2, 3]], \\
\text{(4.9)} & \quad \uparrow A[[1, 2, 3], [3, 4, 5]], \\
\text{(4.10)} & \quad \uparrow A[[2, 3, 4], [2, 3, 4]], \\
\text{(4.11)} & \quad \uparrow A[[3, 4, 5], [1, 2, 3]], \\
\text{(4.12)} & \quad \uparrow A[[3, 4, 5], [3, 4, 5]], \\
\text{(4.13)} & \quad \downarrow A[[1, 2, 3], [2, 3, 4]], \\
\text{(4.14)} & \quad \downarrow A[[2, 3, 4], [1, 2, 3]], \\
\text{(4.15)} & \quad \downarrow A[[2, 3, 4], [3, 4, 5]], \\
\text{(4.16)} & \quad \downarrow A[[3, 4, 5], [2, 3, 4]].
\end{align*}
\]

These pairs create 20 intervals and the following 12 pairs of them satisfy conditions of Lemma 3.3 so they are unconditional:

\[
\begin{align*}
\{\text{(4.8), (4.13)}\}, \{\text{(4.8), (4.14)}\}, \{\text{(4.9), (4.13)}\}, \{\text{(4.9), (4.14)}\}, \{\text{(4.10), (4.13)}\}, \{\text{(4.10), (4.14)}\}, \{\text{(4.11), (4.13)}\}, \{\text{(4.11), (4.14)}\}, \{\text{(4.12), (4.13)}\}, \{\text{(4.12), (4.14)}\},
\end{align*}
\]
The remaining eight pairs are

\[(4.17) \quad \{ \{4,8\}, \{4,15\} \}, \{\{4,8\}, \{4,16\}\}, \{\{4,12\}, \{4,13\}\}, \{\{4,12\}, \{4,14\}\}, \]

and

\[(4.18) \quad \{ \{4,9\}, \{4,13\} \}, \{\{4,9\}, \{4,16\}\}, \{\{4,11\}, \{4,13\}\}, \{\{4,11\}, \{4,15\}\} . \]

Using Lemma 3.1, it is enough to show that only one of the pairs in (4.17) and (4.18) are conditional. In the following matrices $B_1$ and $B_2$, the only conditional pairs of minors are \{\{4,8\}, \{4,15\}\} and \{\{4,9\}, \{4,16\}\}, respectively, and the inequalities obtained from all other seven pairs hold. Thus, these eight pairs of minors are conditional and the conditions obtained from them are independent.

\[
B_1 = \begin{pmatrix} 500 & 300 & 11 & 10 & 1 \\ 8.35 & 10 & 8 & 9 & 1 \\ 6 & 8.61 & x & 8.99 & 1 \\ 2 & 5 & 6 & 60 & 10 \\ 1 & 3 & 4 & 80 & 20 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3.2 & 4 & 5 \\ 1 & 2.95 & x & 12 & 18 \\ 1 & 5 & 16 & 31 & 51 \\ 1 & 7 & 26.25 & 54 & 97 \end{pmatrix} .
\]

The polynomial inequalities obtained from conditional pairs of minors listed in (4.17) and (4.18) are listed below:

\[
\begin{aligned}
&\det A[\{1,2,3\}, \{1,2,3\}](0) < \det A[\{2,3,4\}, \{3,4,5\}](0), \\
&\det A[\{1,2\}, \{1,2\}] \\
&\det A[\{1,2,3\}, \{1,2,3\}](0) < \det A[\{3,4,5\}, \{2,3,4\}](0), \\
&\det A[\{1,2\}, \{1,2\}] \\
&\det A[\{4,5\}, \{4,5\}](0) < \det A[\{1,2,3\}, \{1,2,3\}](0), \\
&\det A[\{4,5\}, \{4,5\}] \\
&\det A[\{1,2,3\}, \{1,2,3\}](0) < \det A[\{2,3,4\}, \{1,2,3\}](0), \\
&\det A[\{1,2\}, \{4,5\}] \\
&\det A[\{1,2,3\}, \{3,4,5\}](0) < \det A[\{2,3,4\}, \{1,2,3\}](0), \\
&\det A[\{1,2\}, \{4,5\}] \\
&\det A[\{1,2,3\}, \{3,4,5\}](0) < \det A[\{2,3,4\}, \{2,3,4\}](0), \\
&\det A[\{1,2\}, \{4,5\}], \\
&\det A[\{1,2,3\}, \{1,2,3\}](0) < \det A[\{2,3,4\}, \{2,3,4\}](0), \\
&\det A[\{1,2\}, \{1,2\}], \\
&\det A[\{3,4,5\}, \{1,2,3\}](0) < \det A[\{2,3,4\}, \{2,3,4\}](0), \\
&\det A[\{4,5\}, \{1,2\}], \\
&\det A[\{3,4,5\}, \{1,2,3\}](0) < \det A[\{2,3,4\}, \{2,3,4\}](0), \\
&\det A[\{4,5\}, \{1,2\}], \\
&\det A[\{3,4,5\}, \{1,2,3\}](0) < \det A[\{2,3,4\}, \{2,3,4\}](0), \\
&\det A[\{4,5\}, \{1,2\}], \\
&\det A[\{3,4,5\}, \{1,2,3\}](0) < \det A[\{2,3,4\}, \{2,3,4\}](0), \\
&\det A[\{4,5\}, \{1,2\}], \\
&\det A[\{3,4,5\}, \{1,2,3\}](0) < \det A[\{2,3,4\}, \{2,3,4\}](0), \\
&\det A[\{4,5\}, \{1,2\}], \\
&\det A[\{3,4,5\}, \{1,2,3\}](0) < \det A[\{2,3,4\}, \{2,3,4\}](0), \\
&\det A[\{4,5\}, \{1,2\}], \\
&\det A[\{3,4,5\}, \{1,2,3\}](0) < \det A[\{2,3,4\}, \{2,3,4\}](0), \\
&\det A[\{4,5\}, \{1,2\}], \\
\end{aligned}
\]
(b) Now, consider the pairs of submatrices containing $x$ where one is of order 2 and the other one has order 3. The only cases for which they do not lie in a 3-by-5 or its transpose submatrix are

\begin{align}
(4.19) \quad & \uparrow A[\{1, 2, 3\}, \{3, 4, 5\}], \downarrow A[\{3, 4\}, \{2, 3\}] \\
(4.20) \quad & \uparrow A[\{3, 4, 5\}, \{1, 2, 3\}], \downarrow A[\{2, 3\}, \{3, 4\}].
\end{align}

Both of these pairs are contained in a 4-by-4 submatrix. The pair of submatrices in (4.19) creates the following condition:

\begin{equation}
\frac{-\det A[\{1, 2, 3\}, \{3, 4, 5\}][0]}{\det A[\{1, 2\}, \{4, 5\}]} < \frac{\det A[\{3, 4\}, \{2, 3\}][0]}{\det A[\{4\}, \{2\}]} = \frac{a_{32}a_{43}}{a_{42}}.
\end{equation}

We show that this inequality is obtained from being partial TP$_3$ and the inequality produced by the pair of submatrices in (4.9) and (4.16), that is

\begin{equation}
\frac{-\det A[\{1, 2, 3\}, \{3, 4, 5\}][0]}{\det A[\{1, 2\}, \{4, 5\}]} < \frac{a_{34}\det A[\{4, 5\}, \{3, 4\}] + a_{43}\det A[\{4, 5\}, \{2, 3\}]}{\det A[\{4, 5\}, \{2, 4\}]}.
\end{equation}

It is enough to show that

\begin{equation}
\frac{a_{32}\det A[\{4, 5\}, \{3, 4\}] + a_{43}\det A[\{4, 5\}, \{2, 3\}]}{\det A[\{4, 5\}, \{2, 4\}]} < \frac{a_{32}a_{43}}{a_{42}}.
\end{equation}

The proof for the inequality (4.21) is similar to that of the inequality (4.7), and is omitted.

Similarly, we can show that the condition obtained from the submatrices in (4.11) and (4.15) and the assumption of partial TP$_3$ imply the condition obtained by (4.20). So the case of 2-by-2 versus 3-by-3 does not create extra independent conditions. Therefore, when $(i, j) = (3, 3)$, there are eight independent conditions needed to ensure the TP$_3$ completability of a 5-by-5 pattern. \hfill \Box

5. TP$_k$ completion of patterns with one unspecified entry. In this section, a characterization of TP$_k$ completable patterns with one unspecified entry, $k \geq 4$, is given.

**Lemma 5.1.** Let $A$ be a $k$-by-$k$ partial TP matrix. For any $r \geq 0$, and $\ell > k$, matrix $A$ can be contained contiguously in a partial TP$_{k+r}$ matrix of order $\ell$. 
Proof. This can be shown by repeatedly using Lemma 2.4 in [4]. □

Lemma 5.2. Let $P$ be an $m$-by-$n$ pattern, $n \geq m \geq 4$, with one unspecified entry in the position $(i, j)$. Suppose $i + j > 4$ and $i + j < m + n - 2$. Then $P$ is not $\text{TP}_k$ completable, for $k \geq 4$.

Proof. We show that there is a partial $\text{TP}_k$ matrix with pattern $P$ and with no $\text{TP}_k$ completion. For this, start with a 4-by-4 partial TP matrix $P$ with no TP completion. There is such a matrix by [4]. Considering the location of $x$ in $P$, and using Lemma 5.1 extend $B$ in the position $(k, k)$ completable, for $k \geq 4$.

In order to find the submatrix $B$ described above, we consider the following cases for $(i, j)$

1. If $i \leq m - 3$ and $j \geq 4$, then $x$ lies in the $(1, 4)$ position of the submatrix $A[(i, i + 1, i + 2, i + 3), (j - 3, j - 2, j - 1, j)]$.
2. If $2 \leq i \leq m$ and $j = 3$, then $x$ lies in the $(2, 3)$ position of the submatrix $A[(i - 1, i, i + 1, i + 2), (1, 2, 3, 4)]$.
3. If $3 \leq i \leq m$ and $j = 2$, then $x$ lies in the $(3, 2)$ position of the submatrix $A[(i - 2, i - 1, i, i + 1), (1, 2, 3, 4)]$.
4. If $4 \leq i \leq m$ and $j = 1$, then $x$ lies in the $(4, 1)$ position of the submatrix $A[(i - 3, i - 2, i - 1, i), (1, 2, 3, 4)]$.
5. If $(i, j) = (m - 2, n - 1), (m - 1, n - 2), (m, n - 3)$, then the submatrix $A[(m - 3, m - 2, m - 1, m), (n - 3, n - 2, n - 1, n)]$ has $(i, j)$ in its off-diagonal entries.
6. If $(i, j) = (m - 2, n - 2), (m - 1, n - 3)$, since $i + j > 4$, then $m + n > 8$, so either $m > 4$ or $n > 4$, in the former case the submatrix $A[(m - 4, m - 3, m - 2, m - 1), (n - 3, n - 2, n - 1, n)]$ has $(i, j)$ in its off-diagonal entries, in the latter case the submatrix $A[(m - 3, m - 2, m - 1, m), (n - 4, n - 3, n - 2, n - 1)]$ has $(i, j)$ in its off-diagonal entries.
7. If $(i, j) = (m - 2, n - 3)$, then again using $i + j > 4$ the smallest possible values for $i$ and $j$ are $(i, j) = (4, 6), (i, j) = (5, 5), (i, j) = (6, 4)$. The submatrices $A[(m - 3, m - 2, m - 1, m), (n - 5, n - 4, n - 3, n - 2)], A[(m - 4, m - 3, m - 2, m - 1), (n - 4, n - 3, n - 2, n - 1)], A[(m - 4, m - 3, m - 2), (n - 3, n - 2, n - 1)]$ contain $x$ on their anti-diagonal entries, respectively.
In each of the above cases, the obtained 4-by-4 partial TP\textsubscript{4} pattern is not TP\textsubscript{4} completable; see [4]. This completes the proof. 

**Lemma 5.3.** Let \( P \) be an \( m \times n \) pattern, \( n \geq m \geq 4 \), with one unspecified entry \( x \) in the position \((i, j)\). Suppose \( i + j \leq 4 \) or \( i + j \geq m + n - 2 \). Then \( P \) is TP\textsubscript{k} completable for \( k \geq 4 \).

**Proof.** Let \( A \) be a partial TP\textsubscript{k} matrix with pattern \( P \). We consider the following cases:

1. If \( i = 1 \), then all of the contiguous minors of order at most \( k \) that contain \( x \) are contained in a \( k \times (k + 3) \) submatrix, say \( B \). Since \( A \) is partial TP\textsubscript{k}, and \( B \) has minors of order less than or equal to \( k \), the submatrix \( B \) is in fact partial TP. The unspecified entry lies in the \((i, j)\) position of \( B \), with \( i + j \leq 4 \), using [4], \( B \) is TP (and therefore TP\textsubscript{k}) completable. Using contiguity, there is a TP\textsubscript{k} completion for \( A \). By similar arguments, it can be shown that there is a TP\textsubscript{k} completion for \( A \) if \( j = 1, i = m, \) and \( j = n \).

2. If \( i = j = 2 \), then the contiguous minors of order at most \( k \) containing \( x \) are contained in a \((k + 1) \times (k + 1)\) submatrix. However, the only minor of order \( k + 1 \) in this submatrix is the determinant of the entire submatrix, and that contains the unspecified entry, so there are no minors of order \( k + 1 \) consisting of only specified entries. Therefore, the submatrix has only minors of order at most \( k \), and because \( A \) is partial TP\textsubscript{k}, the submatrix is partial TP. The unspecified entry lies in the \((2, 2)\) position in this partial TP submatrix, and as \( i + j \leq 4 \), the submatrix has a TP (and so TP\textsubscript{k}) completion. Again using contiguity, there is a TP\textsubscript{k} completion for \( A \). By similar arguments, it follows that there is a TP\textsubscript{k} completion for \( A \) if \( i = m - 1 \) and \( j = n - 1 \).

Lemmas 5.2 and 5.3 imply the following results.

**Theorem 5.4.** For \( m, n, k \geq 4 \), an \( m \times n \) pattern \( P \) with one unspecified entry in the \((i, j)\) position is TP\textsubscript{k} completable if and only if \( i + j \leq 4 \) or \( i + j \geq m + n - 2 \).

**Theorem 5.5.** Let \( P \) be an \( m \times n \) pattern with one unspecified entry. If \( P \) is TP\textsubscript{k} completable, then \( P \) is TP\textsubscript{k-1} completable.

**Proof.** We know that every pattern with exactly one unspecified entry is both TP\textsubscript{1} and TP\textsubscript{2} completable. So the statement is true for \( k < 4 \). For \( k \geq 4 \), the result follows from Theorem 5.4.

**6. Appendix.** Characterizing explicit conditions for TP\textsubscript{k} completability, \( k \geq 4 \), using only upper bounds and lower bounds, requires tedious effort. Here we consider a 7-by-7 partial TP\textsubscript{4} matrix \( A \) where the \((4, 4)\) entry is unspecified. There are sixteen 4-by-4 contiguous minors containing the unspecified entry, eight of which produce...
lower bounds, and eight produce upper bounds. We use the following labelling for the submatrices of \( A \) of order 4 that produce upper bound \((U_i)\) or lower bound \((L_i)\) for \( x \).

\[
L_1 = A[\{1234\}, \{1234\}], \quad L_2 = A[\{1234\}, \{3456\}], \quad L_3 = A[\{2345\}, \{2345\}], \\
L_4 = A[\{2345\}, \{4567\}], \quad L_5 = A[\{3456\}, \{1234\}], \quad L_6 = A[\{3456\}, \{3456\}], \\
L_7 = A[\{4567\}, \{1234\}], \quad L_8 = A[\{4567\}, \{4567\}], \quad U_1 = A[\{1234\}, \{2345\}], \\
U_2 = A[\{1234\}, \{4567\}], \quad U_3 = A[\{2345\}, \{1234\}], \quad U_4 = A[\{2345\}, \{3456\}], \\
U_5 = A[\{3456\}, \{2345\}], \quad U_6 = A[\{3456\}, \{4567\}], \quad U_7 = A[\{4567\}, \{1234\}], \\
U_8 = A[\{4567\}, \{3456\}].
\]

So, there are 64 upper bound/lower bound pairs obtained from only 4-by-4 minors. From these minors 24 of them satisfy conditions of Lemma 3.3 and so they are unconditional pairs of minors. However, the following list of matrices together with Lemma 3.1 show that the remaining pairs of minors are conditional. In addition, there are eight conditions obtained from the TP\(_4\) case. Moreover, there are more pairs of minors (of order 4-by-4 with either 2-by-2 or 3-by-3) that are not known to be conditional or unconditional.

6.1. A list of partial TP\(_4\) matrices with no TP\(_4\) completion.

\[
L_1 > U_2 : \begin{pmatrix}
100 & 12 & 6 & 3 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 3 & 5 & 7 & 9 & 11 & 20 \\
1 & 4 & 9 & x & 20 & 30 & 262.7
\end{pmatrix}.
\]

\[
L_4 > U_3 : \begin{pmatrix}
100 & 12 & 6 & 3 & 1 & 1 & 1 \\
1 & 2 & 3 & 5 & 6 & 7 \\
1 & 3 & 5 & x & 9 & 11 & 20 \\
1 & 4 & 9 & 10 & 20 & 30 & 262.7
\end{pmatrix}.
\]

\[
L_1 > U_4 : \begin{pmatrix}
57 & 12.1 & 5.95 & 3.46 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6.1 \\
1.15 & 3 & 4.998 & 7 & 9 & 11 \\
1.09 & 4 & 9.2 & x & 20 & 30 \\
.999 & 5.4 & 15.49 & 26 & 40 & 200
\end{pmatrix}.
\]
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$L_1 > U_6$:

\[
\begin{pmatrix}
1 & 1.2 & 1 & 1.0099 & 1 & 1 & 1 \\
1 & 4 & 5 & 6.989 & 8 & 9 & 10 \\
1 & 6 & 9 & 16 & 21.0005 & 27 & 35.325 \\
1.000101 & 7 & 12.0925 & x & 38 & 56.01 & 85 \\
1 & 8 & 16 & 38.999 & 69.0001 & 116 & 206 \\
1 & 9 & 22 & 64.55 & 136 & 268.325 & 577
\end{pmatrix}
\]

$L_2 > U_5$:

\[
\begin{pmatrix}
1 & 1 & 1.108 & 1 & 1 & 1 \\
1 & 1.55877 & 2.95047 & 4.41994 & 5.59978 & 5.70515 \\
1 & 1.69127 & 4.15273 & 7.44402 & 10.4814 & 11.3799 \\
1 & 3.20442 & x & 44.1068 & 71.1925 & 84.8527 \\
1 & 3.66803 & 23.3806 & 59.95 & 100.867 & 127.887 \\
1 & 4.16746 & 29.581 & 80.9181 & 144.658 & 206.988
\end{pmatrix}
\]

$L_2 > U_3$:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1.02546 & 1.7817 & 2.02443 & 2.28748 & 3.99454 \\
1 & 1.27716 & 10.6119 & 13.7854 & 17.2935 & 40.8292 \\
1 & 1.57735 & 21.3954 & x & 36.8739 & 103.157 \\
1 & 2.13274 & 42.37 & 56.956 & 81.9643 & 347.708
\end{pmatrix}
\]

$L_2 > U_6$:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1.173 & 1 & 1 & 1 \\
1 & 1.66861 & 2.15911 & 2.54789 & 2.74514 \\
1 & 2.43216 & 3.98167 & 5.85376 & 7.10803 \\
1.0032 & x & 4.35477 & 7.75572 & 10.6744 \\
1 & 2.744 & 4.92001 & 12.5234 & 20.7245 \\
1 & 3.35 & 6.75196 & 77.476 & 176.401
\end{pmatrix}
\]

$L_2 > U_7, L_3 > U_7$:

\[
\begin{pmatrix}
500 & 1 & 1 & 1.01 & 1 & 1 \\
1 & 1.89299 & 2.69541 & 3.06855 & 3.39843 & 4.55957 \\
1 & 2.98368 & 5.64721 & 7.41 & 9.22161 & 15.6336 \\
1 & 3.7812 & 8.21308 & x & 16.775 & 33.0214 \\
1 & 5.65706 & 14.8655 & 28.53 & 46.3549 & 129.497 \\
1 & 7.44993 & 22.0492 & 51.7033 & 97.8835 & 10000000
\end{pmatrix}
\]
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\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1.85217 & 2.2313 & 2.53312 & 2.95498 & 3.29232 \\
1 & 2.983 & x & 11.9684 & 23.1470 & 32.9 \\
1 & 2.98379 & 7.015 & 20.1909 & 45.6336 & 81.9216
\end{pmatrix}
\]

\[
L_3 > U_2 : 
\begin{pmatrix}
1 & 1.2081 & 1.9291 & 2.75604 & 3.01261 & 3.61019 \\
1 & 1.86343 & x & 11.8739 & 14.3995 & 21.3672 \\
1 & 4.05599 & 20.2848 & 49.0769 & 64.9434 & 116.177
\end{pmatrix}
\]

Acknowledgment. All authors were supported in part by NSF Grant #DMS-0751964. This work was done while the last author was a PIMS PDF at the University of Regina.

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