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NON-SINGULAR CIRCULANT GRAPHS AND DIGRAPHS

A.K. LAL† AND A. SATYANARAYANA REDDY‡

Abstract. For a fixed positive integer \( n \), let \( W_n \) be the permutation matrix corresponding to the permutation \( (1 \ 2 \ \cdots \ n^{-1} n \ 2 \ 3 \ \cdots \ n) \). In this article, it is shown that a symmetric matrix with rational entries is circulant if, and only if, it lies in the subalgebra of \( \mathbb{Q}[x]/\langle x^n - 1 \rangle \) generated by \( W_n + W_n^{-1} \). On the basis of this, the singularity of graphs on \( n \)-vertices is characterized algebraically. This characterization is then extended to characterize the singularity of directed circulant graphs. The \( k \)th power matrix \( W_n^k + W_n^{-k} \) defines a circulant graph \( C_k^n \). The results above are then applied to characterize its singularity, and that of its complement graph. The digraph \( C_{r,s,t} \) is defined as that whose adjacency matrix is circulant \( \text{circ}(a) \), where \( a \) is a vector with the first \( r \)-components equal to 1, and the next \( s \), \( t \) and \( n - (r + s + t) \) components equal to zero, one, and zero respectively. The singularity of this digraph (graph), under certain conditions, is also shown to depend algebraically upon these parameters. A slight generalization of these graphs are also studied.

Key words. Graphs, Digraphs, Circulant matrices, Primitive roots.

AMS subject classifications. 05C50, 15A18.

1. Introduction and preliminaries. Let \( \mathbb{Q} \) denote the set of rational numbers and let \( a = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{Q}^n \), for a fixed positive integer \( n \). Consider the shift operator \( T : \mathbb{Q}^n \to \mathbb{Q}^n \), defined by \( T(a_0, a_1, \ldots, a_{n-1}) = (a_{n-1}, a_0, \ldots, a_2) \). The circulant matrix, denoted \( \text{circ}(a) \), associated with \( a \) is the \( n \times n \) matrix whose \( k \)th row is \( T^{k-1}(a) \), for \( k = 1, 2, \ldots, n \). For example, the identity matrix, denoted \( I \), equals \( \text{circ}(1,0,\ldots,0) \), the matrix of all 1’s, denoted \( J \), equals \( \text{circ}(1,1,\ldots,1) \) and \( W_n = \text{circ}(0,1,\ldots,0) \) is the permutation matrix corresponding to the permutation \( (1 \ 2 \ \cdots \ n^{-1} n) \). It is well known that \( W_n \) is the generator of the algebra \( \mathbb{Q}[x]/\langle x^n - 1 \rangle \) (see, for instance, [6]), where \( \mathbb{Q}[x] \) is the polynomial ring over \( \mathbb{Q} \) and \( (x^n - 1) \) denotes the principal ideal of \( \mathbb{Q}[x] \) generated by \( x^n - 1 \). Or equivalently, every circulant matrix of order \( n \) is a polynomial in \( W_n \).

Lemma 1.1. [2] Let \( A \) be an \( n \times n \) matrix with rational entries. Then \( A \) is circulant if, and only if, it is a polynomial over \( \mathbb{Q} \) in \( W_n \).

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For \( a \in \mathbb{Q}^n \), let \( A = \text{circ}(a) \). Then Lemma 1.2 ensures the existence of the polynomial \( \gamma_A(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \in \mathbb{Q}[x] \), called the representor polynomial, with \( A = \gamma_A(W_n) \). One also obtains the following result about circulant matrices.

**Lemma 1.2.** Let \( A = \text{circ}(a) \) be a circulant matrix. Then the eigenvalues of \( A \) equal \( \gamma_A(\zeta_k) = a_0 + a_1 \zeta_k + \cdots + a_{n-1} (\zeta_k)^{n-1} \), for \( k = 0, 1, \ldots, n-1 \), where \( \zeta_n \) denotes a primitive \( n \)-th root of unity.

The reader is referred to [4] and/or [1] for standard concepts and results used in this article without explicit references.

Let \( G = (V, E) \) be a simple graph (has no loops), directed or not, on \( n = |V| \) vertices, where for a finite set \( S \), \( |S| \) denotes the number of elements in \( S \). Given an ordering \( V = \{v_1, v_2, \ldots, v_n\} \) of the vertices, the adjacency matrix of \( A(G) = [a_{ij}] \) of \( G \), is by definition the \( n \times n \) matrix whose \( ij \)-th entry equals the number of edges \( \{v_i, v_j\} \), from the vertex \( v_i \) to the vertex \( v_j \). Observe that, the adjacency matrix \( A(G) \) of a graph \( G \) is symmetric, whereas the adjacency matrix of a directed graph need not be symmetric. The graph \( G \) is said to be nonsingular if \( A(G) \) is a nonsingular matrix and is called circulant if \( A(G) \) is a circulant matrix. For example, the adjacency matrix of the cycle graph \( C_n \) on \( n \) vertices equals \( \text{circ}(0, 1, 0, \ldots, 1) = W_n + W_n^{-1} \) and \( \gamma(x) = x + x^{n-1} \) is its representor polynomial. Thus, using Lemma 1.2 the graph \( C_n \) is singular if, and only if, \( n \) is a multiple of 4.

Recall that the \( n \)-th cyclotomic polynomial, denoted \( \Phi_n(x) \), is an element of \( \mathbb{Z}[x] \) and equals \( \prod_{\substack{d|n \leq k \leq n}} (x - \zeta_n^k) \). Also, \( x^n - 1 = \prod_{d|n} \Phi_d(x) \) (here \( a \mid b \) means \( a \) ‘divides’ \( b \)). Moreover, \( \Phi_n(x) \) is the minimal polynomial of \( \zeta_n \) and hence, \( f(\zeta_n) = 0 \) for some \( f(x) \in \mathbb{Z}[x] \) implies \( \Phi_n(x) \) divides \( f(x) \in \mathbb{Z}[x] \). The next result appears in [5].

**Lemma 1.3.** [5] Let \( p \) be a prime and let \( n \) be a positive integer. Then

\[
\Phi_{pn}(x) = \begin{cases} 
\Phi_n(x^p), & \text{if } p \mid n, \\
\Phi_n(x), & \text{if } p \nmid n.
\end{cases}
\]

In particular, for each positive integer \( k \), \( \Phi_{pk}(x) = 1 + x^{p-1} + x^{2(p-1)} + \cdots + x^{(p-1)p^{k-1}}. \)

The following result is an application of Lemma 1.2. This result also appears in [5] (also see Corollary 10 in [6]).

**Lemma 1.4.** [5] [6] Let \( A = \text{circ}(a) \). Then \( A \) is singular if, and only if, \( \deg(\gcd(\gamma_A(x), x^n - 1)) \geq 1 \).

Lemma 1.4 gives us the following useful remarks.
Remark 1.5.

1. Let $A = \text{circ}(a)$ with $a = (1, \ldots, 1, 0, \ldots, 0)$. Then $A$ is singular if, and only if, $\gcd(s, n) > 1$.

2. Let $A = \text{circ}(a)$ with $a_i = 0$, for $0 \leq i \leq k - 1$ and $a_k \neq 0$. Then $\gamma_A(x) = x^k \Gamma_A(x)$, for some polynomial $\Gamma_A(x) \in \mathbb{Q}[x]$. Thus, using Lemma 1.4, $A$ is singular (nonsingular) if, and only if, $W^n_k A$ is singular (nonsingular), for some $k = 0, 1, \ldots, n - 1$. That is, to study singularity (nonsingularity) of $A$, it is enough to study $\Gamma_A(x)$.

Using Remark 1.5.2 and Lemma 1.4, the following result is immediate, and hence, the proof is omitted.

Lemma 1.6. Let $G$ be a circulant digraph of order $n$. Then $G$ is singular if, and only if, $\Phi_d(x) \mid \Gamma_A(G)(x)$, for some divisor $d \neq 1$ of $n$.

As an immediate corollary of Lemma 1.6 and 1.3, the following result follows.

Corollary 1.7. Let $p$ be a prime and let $k, \ell$ be positive integers with $p \nmid k$. Also, let $G$ be a $k$-regular circulant graph/digraph on $p^\ell$ vertices. Then $G$ is nonsingular.

Proof. Using Lemma 1.6, it is enough to show that $\Phi_d(x) \nmid \Gamma_A(G)(x)$, for every $d \mid p^\ell$ with $d \neq 1$. On the contrary, assume that $\Gamma_A(G)(x) = \Phi_d(x)g(x)$, for some $g(x) \in \mathbb{Z}[x]$. Then using Lemma 1.3, $\Phi_d(1) = p$, for every $d \mid p^\ell$ with $d \neq 1$. As $g(x) \in \mathbb{Z}[x]$, $g(1) \in \mathbb{Z}$ and hence $k = \Gamma_A(G)(1) = \Phi_d(1)g(1) = p g(1)$. A contradiction to the assumption that $p \nmid k$.

The remaining part of this paper consists of two more sections that are mainly concerned with applications of Lemma 1.6. Section 2 gives necessary and sufficient conditions for a few classes of circulant graphs to be nonsingular and Section 3 gives possible generalization of the results studied in Section 2. Before proceeding to Section 2 recall that for a graph $G = (V, E_1)$, the complement graph of $G$, denoted $G^c = (V, E_2)$, is a graph in which $(u, v) \in E_2$ whenever $(u, v) \notin E_1$ and vice versa, for every $u \neq v \in V$. Note that as $G$ is a simple graph, $G^c$ is also simple and $A(G^c) = J - A(G) - I$. Moreover, $G$ is circulant if, and only if, $G^c$ is circulant.

2. Some singular circulant graphs. We start this section by showing that the adjacency matrix of a circulant graph on $n$ vertices is a polynomial in the adjacency matrix of $C_n$.

For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, let $C_n^i$ denote a graph whose vertex set is same as the vertex set of the cycle graph $C_n$ and $(x, y)$ is an edge in $C_n^i$ whenever the distance of $x$ from $y$ in $C_n$ is exactly $i$. Also, define the matrices $A_i$ (commonly called the distance matrices
of $C_n$ by

$$A_i = W_n^i + W_n^{n-i} \quad \text{and} \quad A_{\frac{n}{2}} = \begin{cases} W_n^\frac{n}{2}, & \text{if } n \text{ is even,} \\ W_n^\frac{n}{2} + W_n^{n-\frac{n}{2}}, & \text{if } n \text{ is odd.} \end{cases}$$

Then note that $A_i = A(C_n^i)$, for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

The identity $(x^k + x^{-k}) = (x + x^{-1})(x^{k-1} + x^{1-k} - x^{k-2} + x^{2-k})$ enables us to readily establish, by mathematical induction, that $x^k + x^{-k}$ is a monic polynomial in $x + x^{-1}$ of degree $k$. Consequently, $A_i$’s, for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, are polynomials of degree $\leq i$, in the adjacency matrix of $C_n$ over $\mathbb{Q}$. Thus, every circulant symmetric matrix with rational entries lies in the subalgebra of $\mathbb{Q}[x]/(x^n - 1)$ generated by $W_n + W_n^{-1}$ (for $n$ even, note that $2W_n^\frac{n}{2} = W_n^\frac{n}{2} + W_n^{n-\frac{n}{2}}$).

**Theorem 2.1.** Let $B = \text{circ}(q)$, with $q \in \mathbb{Q}^n$. Then $B$ is a symmetric if, and only if, $B$ is a polynomial in $W_n + W_n^{-1}$ with rational coefficients.

**Proof.** One part is immediate, as every polynomial in $W_n + W_n^{-1}$ is a symmetric circulant matrix.

Conversely, let $B = \text{circ}(q)$ be symmetric. Then $B = \sum_{i=0}^{n-1} q_i W_n^i$ and $B' = \sum_{i=0}^{n-1} q_i W_n^{n-i}$. But $B = B'$ implies $q_i = q_{n-i}$, for $1 \leq i \leq n-1$. Thus, $B = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} q_i A_i$.

Observe that for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $\gamma_{Ai}(x) = x^i (1 + x^{n-2i})$ and $\gamma_{Ai}(x) = x^{i+1} + x^{i+1}$, if $n$ is even, and $\gamma_{Ai}(x) = x^i x^{n-2i} (1 + x)$, if $n$ is odd. Using these, the next result gives a necessary and sufficient condition for the graphs $C_n^i$, for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, to be singular.

**Proposition 2.2.** Let $n \geq 3$ and $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Then the graph $C_n^i$ is singular if, and only if, $n$ is a multiple of $4$ and $\gcd(i, \frac{n}{2}) = \frac{n}{4}$.

**Proof.** Note that $\Gamma_{Ai}(x) = 1 + x$ if $n$ is odd and $\Gamma_{Ai}(x) = 1$ if $n$ is even. Hence, $A_i$ is nonsingular for all $n$. For $1 \leq i < \lfloor \frac{n}{2} \rfloor$, $\Gamma_{Ai}(x) = 1 + x^{n-2i}$ and hence $A_i$ is singular if, and only if, $\Gamma_{Ai}(\zeta_n^k) = 0$, for some $k, 1 \leq k \leq n - 1$. That is, one needs $(\zeta_n^k)^{n-2i} = -1$. This holds if, and only if, $4$ divides $n$ and $\gcd(i, \frac{n}{2}) = \frac{n}{4}$.

As an immediate consequence of Proposition 2.2, the graph $C_n$ is singular if, and only if, $4 \mid n$. The next result gives a necessary and sufficient condition for the complement graph $(C_n^i)^c$, of $C_n^i$, to be singular.

**Proposition 2.3.** Let $n \geq 4$. Then the graph

1. $(C_n^\lfloor \frac{n}{2} \rfloor)^c$ is singular if, and only if, $n$ is even or $n \equiv 3 \pmod{6}$.
2. $(C_n^i)^c$, for $1 \leq i < \lfloor \frac{n}{2} \rfloor$, is singular if, and only if, $3 \nmid n$ and $\gcd(i, n) = \frac{n}{3}$.
Proof. Note that $A = A((C_{n}^{0})^{r}) = J - A_{1} + I$ and $\gamma_{A}(x) = \frac{x^{n} - 1}{x - 1} - (1 + x^{1/2})$,
for $n$ even. Thus, using Lemma 1.2 the graph $(C_{n}^{0})^{c}$ is singular, whenever $n$ is even. And for $n$ odd, $\gamma_{A}(x) = \frac{x^{n} - 1}{x - 1} - (1 + x^{1/2} + x^{1/2+1})$. Consequently, $(C_{n}^{0})^{c}$
is singular if, and only if, $1 + (\zeta_{2}^{k})^{1} + (\zeta_{2}^{k})^{1} = 0$, for some $k$, $1 \leq k \leq n - 1$. Or equivalently, $\zeta_{n}^{1/2}$ is a primitive 3-rd root of unity. Thus, $n \equiv 3 \pmod{6}$.

Now assume that $1 \leq i < \lfloor \frac{n}{2} \rfloor$. Then $A((C_{n}^{i})^{r}) = J - A_{1} - I$, and hence,
$\gamma_{A(C_{n}^{i})}(x) = \frac{x^{n} - 1}{x - 1} - (1 + x^{i} + x^{n-i})$. Thus, $(C_{n}^{i})^{c}$ is singular if, and only if, $1 + \zeta_{n}^{k} + \zeta_{n}^{-k} = 0$, for some $k$, $1 \leq k \leq n - 1$. That is, $\zeta_{n}^{k}$ is a primitive 3-rd root of unity. Thus, $(C_{n}^{i})^{c}$ is singular if, and only if, $3 | n$ and $\gcd(i, n) \not\equiv 0$. \square

As an immediate consequence of Proposition 2.3 it follows that the complement graph $C_{n}^{c}$, of $C_{n}$, is singular if, and only if, $3 | n$.

We now obtain necessary and sufficient conditions for nonsingularity of circulant graphs that appeared in [9]. For the sake of clarity, his notations have been modified. Fix a positive integer $n \geq 3$ and let $1 \leq r < \lfloor \frac{n}{2} \rfloor$. The first class of circulant graphs, denoted $C_{n}^{(r)}$, has the same vertex set as the vertex set of the cycle $C_{n}$ and $(x, y)$ is an edge whenever the length of the shortest path from $x$ to $y$ in $C_{n}$ is at most $r$. Note that $C_{n}^{(1)}$ is the complete graph. The second class of graphs, denoted $C(2n, r)$, is a graph on $2n$ vertices and its adjacency matrix is the sum of the adjacency matrices of $C_{2n}^{(r)}$ and $C_{2n}^{(r)}$, where $1 \leq r < n$. A necessary and sufficient condition for the graph $C(2n, r)$ to be singular appears later in Theorem 2.4. A similar result on $C_{n}^{(r)}$ appears as Theorem 2.2 of [9]. A separate proof is given for the sake of completeness.

Theorem 2.4. ([9] Let $n \geq 3$ and let $1 \leq r < \lfloor \frac{n}{2} \rfloor$. Then the graph $C_{n}^{(r)}$ is singular if, and only if, one of the following conditions hold:

1. $\gcd(n, r) > 1$,
2. $\gcd(n, r) = 1$, $n$ is even and $\gcd(r + 1, n)$ divides $\frac{n}{2}$.

Proof. Note that $A = A(C_{n}^{(r)}) = \text{circ}(0, 1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 1)$ and hence
$(x - 1)\Gamma_{A}(x) = (x^{r} - 1)(1 + x^{n-r-1})$. Therefore, $C_{n}^{(r)}$ is singular if, and only if, $\Gamma_{A}(\zeta_{n}^{d}) = 0$, for some $d$, $1 \leq d \leq n - 1$. Or equivalently either $(\zeta_{n}^{d})^{r} - 1 = 0$ or $1 + (\zeta_{n}^{d})^{n-r-1} = 0$.

But $(\zeta_{n}^{d})^{r} - 1 = 0$ whenever $\gcd(r, n) > 1$. If $\gcd(r, n) = 1$ then $1 + (\zeta_{n}^{d})^{n-r-1} = 0$ and this implies that $n$ is even and $\gcd(r + 1, n)$ divides $\frac{n}{2}$. \square
The following result follows from Remark [15, 16]

**Corollary 2.5.** Let \( n \geq 3 \) and let \( 1 \leq r < \lfloor \frac{n}{2} \rfloor \). Then the graph \((C_n^r)^c\) is nonsingular if, and only if, \( \gcd(n, 2r + 1) = 1 \).

**Proof.** Note that \( A = A((C_n^r)^c) = \text{circ}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0) \) and hence \((x - 1)A(x) = x^{n-2r-1} - 1\). Thus, using Remark [15], \((C_n^r)^c\) is singular if, and only if, \( \gcd(2r + 1, n) > 1 \).

Before proceeding with the next result that gives a necessary and sufficient condition for the graph \( C(2n, r) \) to be singular, we state a result that appears as Proposition 1 in Kurshan & Odlyzko [7]

**Lemma 2.6.** Let \( m \) and \( n \) be positive integers with \( m \neq n \) and let \( \zeta_n \) be a primitive root of unity. Then there exists a unit \( u \in \mathbb{Z}[\zeta_n] \) dependent on \( m, n \) and \( \zeta_n \) such that

\[
\Phi_m(\zeta_n) = \begin{cases} 
pu, & \text{if } \frac{m}{n} = p^n, \text{ a prime, } \alpha > 0; \\
(1 - \zeta^{p^n})u, & \text{if } \frac{m}{n} = p^{-\alpha}, \text{ a prime, } \alpha > 0; \ p \mid m; \\
(1 - \zeta^{p^n+1})^{ho-1}u, & \text{if } \frac{m}{n} = p^{-\alpha}, \text{ a prime, } \alpha > 0; \ p \mid m; \\
u, & \text{otherwise.} 
\end{cases}
\]

**Theorem 2.7.** Let \( n \) and \( r \) be positive integers such that the circulant graph \( C(2n, r) \) is well defined. Then the circulant graph \( C(2n, r) \) is singular if, and only if, \( \gcd(n, 2r + 1) \geq 3 \).

**Proof.** Note that \( A = A(C(2n, r)) = \text{circ}(0, 1, \ldots, 1, 0, 1, \ldots, 1, 0, \ldots, 1) \).

Consequently, \( \gamma_A(x) = x + x^2 + \cdots + x^r + x^n + x^{2n-r} + \cdots + x^{2n-1} = x\Gamma_A(x) \) and \( (x - 1)\Gamma_A(x) = x^r - 1 + x^{n-1}(x - 1) + x^{2n-r-1}(x^r - 1) \)

\[
= x^r(1 - x^{2n-2r-1}) + (x^n - 1) - (x^{n-1} - x^{2n-1}) \\
= (x^n - 1)(x^{n-1} + 1) - x^r(x^{2n-2r-1} - 1).
\]

Now, let us assume that \( \gcd(n, 2r + 1) = d \geq 3 \). Then \((\zeta_{2n/d}^{2n/d}) = 0 \) as

\[
\left(\zeta_{2n/d}^{2n/d}\right)^n = \left(\zeta_{2n/d}^{2n/d}\right)^{n/d} = 1 = \left(\zeta_{2n/d}^{2n/d}\right)^{2r+1/d} = \left(\zeta_{2n/d}^{2n/d}\right)^{2n-2r-1}. 
\]

Hence, the circulant graph \( C(2n, r) \) is singular.

Conversely, let us assume that the graph \( C(2n, r) \) is singular. This implies that there exists an eigenvalue of \( C(2n, r) \) that equals zero. That is, there exists a \( k, 1 \leq k \leq 2n - 1 \), such that \( \gamma_A(\zeta_{2n}^k) = 0 \). We will now show that if \( \gcd(n, 2r + 1) = 1 \)
then the expression \((x - 1)\Gamma_{A}(x)\) evaluated at \(x = \zeta_{2n}^{k}\) can never equal zero, for any \(k, 1 \leq k \leq 2n - 1\), and this will complete the proof of the result.

We need to consider two cases depending on whether \(k\) is odd or \(k\) is even. Let \(k = 2m,\ 1 \leq m < n\). Then, using \(\gcd(n, 2r + 1) = 1, (\zeta_{2n}^{2m} - 1)\Gamma_{A}(\zeta_{2n}^{2m})\) equals
\[-(\zeta_{2n}^{2m})^{r} \left( [ (\zeta_{2n}^{2m})^{-(2r + 1)} - 1 ] \right) \neq 0.\]

Now, let \(k = 2m + 1, 0 \leq m \leq n - 1\). Then \((\zeta_{2n}^{2m+1} - 1)\Gamma_{A}(\zeta_{2n}^{2m+1})\) equals
\[
\left( (\zeta_{2n}^{2m+1})^{n} - 1 \right) \left( (\zeta_{2n}^{2m+1})^{(n-1)} + 1 \right) - (\zeta_{2n}^{2m+1})^{r} \left( (\zeta_{2n}^{2m+1})^{2n - 2r - 1} - 1 \right)\\
= -2 \left( \frac{\zeta_{2n}^{2m+1} - 1}{\zeta_{2n}^{2m+1}} \right) - \zeta_{2n}^{2m+1} + 1 - \frac{(\zeta_{2n}^{2m+1})(2r+1)}{\zeta_{2n}^{2m+1} - 1} - \frac{(\zeta_{2n}^{2m+1})(2r+1)}{\zeta_{2n}^{2m+1} - 1}\\
= -2 \zeta_{2n}^{2m+1} + \frac{(\zeta_{2n}^{2m+1})(2r+1)}{\zeta_{2n}^{2m+1} - 1} - \frac{(\zeta_{2n}^{2m+1})(2r+1)}{\zeta_{2n}^{2m+1} - 1}\\
= -(\zeta_{2n}^{2m+1} - 1) \left[ -2 \zeta_{2n}^{2m+1} + \frac{(\zeta_{2n}^{2m+1})(2r+1)}{\zeta_{2n}^{2m+1} - 1} - \frac{(\zeta_{2n}^{2m+1})(2r+1)}{\zeta_{2n}^{2m+1} - 1} \right]\\
\tag{2.1}
\end{align*}
\]

Note that, \(\zeta_{2n}^{2m+1}\) is a \(d\)-th primitive root of unity, for some \(d\) dividing \(2n\). As \(\gcd(2r + 1, 2n) = 1, \gcd(2r + 1, d) = 1\) and hence using Lemma \(2.6\), \(\prod_{\ell|(2r+1), \ell \neq 1} \Phi_{\ell}(\zeta_{2n}^{2m+1})\) is a unit in \(\mathbb{Z}[\zeta_{d}]\). That is, \(\prod_{\ell|(2r+1), \ell \neq 1} \Phi_{\ell}(\zeta_{2n}^{2m+1}) = 1\). Hence, in equation \(2.1\), the term in the parenthesis cannot be zero. Thus, the result for the odd case as well.

Note that the necessary part of Theorem \(2.7\) was stated and proved as Theorem \(2.1\) in \(\cite{9}\). We will now try to understand the complement graph \(C(2n, r)^{c}\) of \(C(2n, r)\).

**Proposition 2.8.** Let \(n\) and \(r\) be positive integers such that the circulant graph \(C(2n, r)\) is well defined. Then \(C(2n, r)^{c}\) is nonsingular if, and only if,

1. \(n\) and \(r\) have the same parity,
2. \(\gcd(n, r + 1) = 1\), and
3. the highest power of \(2\) dividing \(n\) is strictly smaller than the highest power of \(2\) dividing \(n - r\).

**Proof.** Note that \(A = A(C(2n, r)^{c}) = \text{circ}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 0)\).

Thus, \((x - 1)\Gamma_{A}(x) = (1 + x^{n-r})(x^{n-r-1} - 1)\). Let us now assume that the graph \(C(2n, r)^{c}\) is nonsingular. That is, \(\Gamma_{A}(\zeta_{2n}^{k/d}) \neq 0\), for any \(k = 1, 2, \ldots, 2n - 1\).

Note that if \(n\) and \(r\) have opposite parity then \(\gcd(2n, n - r - 1) = d \geq 2\) and hence \(\Gamma_{A}(\zeta_{2n}^{2/d}) = 0\). Also, if \(n\) and \(r\) have the same parity and \(\gcd(n, r + 1) = d \geq 2\) then \(n - r - 1\) is odd and \(\gcd(2n, n - r - 1) = \gcd(n, n - r - 1) = \gcd(n, r + 1) = d\).
Hence, in this case again, $\Gamma_A(\zeta_{2n/d}^r) = 0$.

Now, the only case that we need to check is “$n$ and $r$ have the same parity, $\gcd(n, r + 1) = 1$ and the highest power of $2$ dividing $n$ is greater than or equal to the highest power of $2$ dividing $n - r$”.

As $n$ and $r$ have the same parity and $\gcd(n, r + 1) = 1$, we get $\gcd(2n, n - r - 1) = 1$ and thus $(\zeta_{2n}^n)^{n-r-1} - 1 \neq 0$, for any $k = 1, 2, \ldots, 2n - 1$. Therefore, we need to find a condition on $k$ so that $1 + (\zeta_{2n}^n)^{n-r-1} - 1 = 0$. But this is true if, and only if, $\gcd(2n, n - r) | n$, or equivalently, the highest power of $2$ dividing $n$ is greater than or equal to the highest power of $2$ dividing $n - r$.

**Remark 2.9.** Observe that using Proposition 2.8, the graph $C(2n, r)$ is non-singular, whenever $n$ and $r$ are both odd and $\gcd(n, r + 1) = 1$. Such numbers can be easily computed. For example, a class of such graphs can be obtained by choosing two positive integers $s$ and $t$ with $s > t$ and defining $n = 2^s - 2^t + 1$ and $r = 2^t - 1$.

3. Generalizations. In this section, we look at a few classes of graphs/digraphs, which are generalizations of the graphs that appear in Section 2. We first start with a class of circulant digraphs.

Let $A = \text{circ}(q)$, where $q = (1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$, for certain non-negative integers $r, s, t$ and $n$. Let the corresponding digraphs be denoted $C_{r,s,t}$. Then a few conditions under which the $C_{r,s,t}$ digraphs are singular appears next.

**Proposition 3.1.** The $C_{r,s,t}$ digraph is singular if

1. $\gcd(n, t, r) > 1$, or
2. $\gcd(n, t) = 1$ and one of the following condition holds:
   (a) there exists $d \geq 2$ such that $d | s$ and $t = \ell r$, for some positive integer $\ell \equiv -1 \pmod{d}$.
   (b) $n$ is even, there exists an even integer $d$ such that $(r + s)$ is an odd multiple of $\frac{d}{2}$ and $t = \ell r$, for some positive integer $\ell \equiv 1 \pmod{d}$.

**Proof.** Part 1. Observe that if $A = A(C_{r,s,t})$ then

\[ (x - 1)\gamma_A(x) = (x^r - 1) + x^t(x^t - 1). \]

If $\gcd(n, r, t) = k > 1$, then $\zeta_n^{n/k}$ is a root of equation (3.1), and hence, the $C_{r,s,t}$ digraph is singular. This completes the proof of the first part.

Part 2. Let $\gcd(n, t) = 1$ and suppose that there exists a positive integer $d \geq 2$ such that $d | s$ and $t = \ell r$, for some positive integer $\ell \equiv -1 \pmod{d}$. So, there exists $\beta \in \mathbb{Z}$ such that $\ell = \beta d - 1$. In this case, the RHS of equation (3.1) evaluated at $\zeta_{2n/d}^r$ reduces to $0$. Thus, the required result holds.
Part 2b: Let \( \gcd(n, t) = 1 \) and \( n = 2m \). Also, suppose that there exists an even positive integer \( d \) such that \( r + s \) is an odd multiple of \( \frac{d}{2} \) and \( t = \ell r \), for some positive integer \( \ell \equiv 1 \pmod{d} \). Then there exists \( \beta \in \mathbb{Z} \) such that \( \ell = \beta d + 1 \). In this case, the RHS of equation (3.1) evaluated at \( \zeta_n^{n/d} \) reduces to \((\zeta_n^{rn/d} - 1) \left(1 + \zeta_n^{n/2}\right)\). Thus, under the given conditions, the corresponding digraph \( C_{r,s,t} \) is singular.

Hence, the proof of the lemma is complete. 

![Fig. 3.1. Singular circulant \((2, 1, 2)\)-graph on 8 vertices and \((2, 3, 2)\)-graph on 10 vertices.](image)

Thus, the above result gives conditions under which the \( C_{r,s,t} \)-digraphs, for non-negative values of \( r, s \) and \( t \), are singular. For certain values of \( r, s \) and \( t \), these digraphs correspond to graphs. For example, for a fixed positive integer \( r \), the

1. \( C_{r,1,r} \)-digraph is singular, whenever \( n \) is chosen so that \( \gcd(n, r) \geq 2 \) and \( n \geq 2r + 1 \). Using Remark 1.5.2, one obtains a circulant graph with adjacency matrix \( \text{circ}(0, 1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 1) \). In particular, for \( r = 2 \) and \( n = 8 \), the \( C_{2,1,2} \)-graph shown in Figure 3.1 is singular.

2. \( C_{2r,2r+1,2r} \)-digraph is singular, whenever \( n \) is even and \( n \geq 6r + 1 \). In this case, \( \text{circ}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0) \) is the adjacency matrix of a circulant graph. In particular, for \( r = 1 \) and \( n = 10 \), the \( C_{2,3,2} \)-graph shown in Figure 3.1 is singular.

We now define another class of circulant digraphs and obtain conditions under which the circulant digraphs are singular. Let \( i, j, k \) and \( \ell \) be non-negative integers such that \( j > \ell \) and \( kj + i + \ell < n \) and let \( C_{i,j,k,\ell} \) be a circulant digraph on \( n \) vertices with \( \gamma_A(C_{i,j,k,\ell})(x) = \sum_{t=0}^{n} x^{t+j} = x^i \prod_{s|t+1, s \neq 1} \Phi_s(x) \cdot \prod_{t|(k+1)j, t \neq \ell} \Phi_{t}(x) \) as its representer polynomial. Thus, we have the following theorem which we state without proof.
Non-singular Circulant Digraphs

**Theorem 3.2.** The circulant digraph $C_{i,j,k,\ell}$, defined above, is singular if, and only if, either $\gcd(\ell + 1, n) \geq 2$ or $\gcd\left(k + 1, \frac{n}{\gcd(n,j)}\right) \geq 2$.

**Remark 3.3.** Note that we can vary the non-negative integers $i, j, k$ and $\ell$ to define quite a few classes of circulant digraphs. For example, the graphs $G(r,t)$ in [3] are a particular case of $C_{i,j,k,\ell}$ and Theorem 3.2 is a generalization of Remark 1.5.1.

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**REFERENCES**