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## THE ALGEBRAIC CONNECTIVITY OF GRAPHS WITH GIVEN STABILITY NUMBER\*

SHUNZHE ZHANG<sup>†</sup>, QIN ZHAO<sup>‡</sup>, AND HUIQING LIU<sup>§</sup>

**Abstract.** In this paper, the authors investigate the algebraic connectivity of connected graphs, and determine the graph which has the minimum algebraic connectivity among all connected graphs of order  $n$  with given stability number  $\alpha \geq \lceil \frac{n}{2} \rceil$ , or covering number, respectively.

**Key words.** Graph, Laplacian matrix, Algebraic connectivity, Stability number.

**AMS subject classifications.** 05C50, 15A18.

**1. Introduction.** Let  $G = (V(G), E(G))$  be a simple undirected graph with  $n$  vertices. Two distinct adjacent vertices are *neighbors*, the set of neighbors of a vertex  $v$  in  $G$  is denoted by  $N_G(v)$ . The *degree* of a vertex  $v \in V(G)$ , denoted by  $d_G(v)$  or simply  $d(v)$ , is the number of edges of  $G$  incident with  $v$ . A *pendant vertex* is a vertex of degree 1. A *pendant neighbor* is the vertex adjacent to a pendant vertex, let  $PV(G)$  and  $PN(G)$  be the vertex set of all pendant vertices and all pendant neighbors of  $G$ , respectively. A *pendant star* of a tree  $T$  is a maximal subtree of  $T$  induced on pendant vertices together with the pendant neighbor to which they are attached. The *distance* between any two vertices is the number of edges in a shortest path joining them, the *diameter* of a graph  $G$  is the greatest distance between any two vertices of  $G$ . Let  $P_n$  and  $K_{1,n-1}$  denote the path and the star of order  $n$ , respectively.

A *stable set* of a graph  $G$  is a set of vertices no two of which are adjacent. A stable set is *maximum* if the graph contains no larger stable set, the cardinality of a maximum stable set in a graph  $G$  is called the *stability number* of  $G$  and is denoted by  $\alpha(G)$ . A *covering* of a graph is a set of vertices which meet all edges of the graph, the minimum cardinality of a graph  $G$  is called the *covering number* of  $G$ , denoted by  $\beta(G)$ . For a graph  $G$  of order  $n$ ,  $\alpha(G) + \beta(G) = n$ .

Let  $G$  and  $H$  be two disjoint graphs. The *disjoint union* of  $G$  and  $H$ , denoted by  $G \cup H$ , is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ . The *coalescence* of  $G$  and  $H$ , denoted by  $G(u) \diamond H(v)$  or  $G(w) \diamond H(w)$ , is obtained from  $G$  and  $H$  by identifying one vertex  $u \in V(G)$  with one vertex  $v \in V(H)$  and forming a new vertex  $w$ .

For a graph  $G$  with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$ , let  $A(G) = (a_{ij})$  be the adjacency matrix of  $G$ , where  $a_{ij} = 1$  if  $v_i$  is adjacent to  $v_j$ , and  $a_{ij} = 0$ , otherwise. And  $D(G) = \text{diag}(d(v_1), d(v_2), \dots, d(v_n))$  be the diagonal matrix of vertex degrees. The matrix  $L(G) = D(G) - A(G)$  is called the *Laplacian matrix* of

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$G$ . It is known that  $L(G)$  is real and positive semidefinite and its eigenvalues can be arranged as:

$$\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G) = 0.$$

The smallest eigenvalue of  $L(G)$  is zero with the vector of all ones as its eigenvector, it has multiplicity one if and only if  $G$  is connected. The second smallest eigenvalue of  $L(G)$  is positive if and only if  $G$  is connected, it is also denoted by  $\mu(G)$ , and is called the *algebraic connectivity* of  $G$ . The eigenvectors corresponding to  $\mu(G)$  are usually called the *Fiedler vectors* of  $G$  (see [4]). A graph is called the *minimizing graph* in a class of graphs if its algebraic connectivity attains the minimum among all graphs in the class.

For a graph  $G$ , let  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ , and  $V(G) = (v_1, v_2, \dots, v_n)$ . Then  $\mathbf{x}$  can be considered as a function defined on  $G$ , each vertex  $v_i$  mapped to  $x_i = x(v_i)$ . If  $\mathbf{x}$  is an eigenvector of  $L(G)$ , then  $x(v)$  is the entry of  $\mathbf{x}$  corresponding to  $v$ ,  $v \in V(G)$ . Then

$$\mathbf{x}^T L(G) \mathbf{x} = \sum_{uv \in E(G)} [x(u) - x(v)]^2. \quad (1.1)$$

In addition, for an arbitrary unit vector  $\mathbf{x} \in \mathbb{R}^n$  orthogonal to  $\mathbf{1}$ ,  $\mu(G) \leq \mathbf{x}^T L(G) \mathbf{x}$  with equality if and only if  $\mathbf{x}$  is a Fiedler vector of  $G$ .

There are many results on the algebraic connectivity, see [1]–[6], [9], [10], [12] and [13]. Fallat and Kirkland [6] have determined the unique (up to isomorphism) trees that maximize and minimize the algebraic connectivity over all trees of order  $n$  with specified diameter. Fallat, Kirkland and Pati [7] discussed the graph that minimizes the algebraic connectivity over all connected graphs of order  $n$  with fixed girth. In [10], Kirkland presented a bound on the algebraic connectivity of a graph in terms of the number of cut points. In [11], Lal, Patra and Sahoo have given some results about the algebraic connectivity with fixed number of pendant vertices. Fan and Tan [8] obtained some lower bounds for the algebraic connectivity with given domination number. Xu, Fan and Tan [15] determined the lower bounds for the algebraic connectivity in terms of matching number or edge covering number. In this paper, we characterize the unique graph whose algebraic connectivity is minimum among all connected graphs of order  $n$  with given stability number or covering number, respectively.

**2. Lemmas.** In this section, we give some lemmas used in the proof of our results.

LEMMA 2.1. [5] *Let  $T$  be a tree with a Fiedler vector  $\mathbf{x}$ . Then exactly one of the two cases occurs:*

*Case A. All values of  $x$  are nonzero. Then  $T$  contains exactly one edge  $pq$  such that  $x(p) > 0$  and  $x(q) < 0$ . The values in vertices along any path in  $T$  which starts in  $p$  and does not contain  $q$  strictly increase, the values in vertices along any path starting in  $q$  and not containing  $p$  strictly decrease.*

*Case B. The set  $N_0 = \{v : x(v) = 0\}$  is non-empty. Then the graph induced by  $N_0$  is connected and there is exactly one vertex  $z \in N_0$  having at least one neighbor not belonging to  $N_0$ . The values along any path in  $T$  starting in  $z$  are strictly increasing, or strictly decreasing, or zero.*

If Case B in Lemma 2.1 occurs, then the vertex  $z$  is called the *characteristic vertex*, and  $T$  is called a *Type I tree*; otherwise,  $T$  is called a *Type II tree* in which case the edge  $pq$  is called the *characteristic edge*. The characteristic vertex or characteristic edge of a tree is independent of the choice of Fiedler vectors; see [12].

LEMMA 2.2. [8] *Let  $G_1$  be a connected graph containing at least two vertices  $v_1, v_2$ , and let  $G_2$  be a nontrivial connected graph containing a vertex  $u$ . Let  $G = G_1(v_2) \diamond G_2(u)$  and  $G^* = G_1(v_1) \diamond G_2(u)$ . If there*

exists a Fiedler vector  $\mathbf{x}$  of  $G$  such that  $x(v_1) \geq x(v_2) \geq 0$  and all vertices in  $G_2$  are nonnegatively valued by  $x$ , then  $\mu(G^*) \leq \mu(G)$ , with equality if and only if  $x(v_1) = x(v_2) = 0$ ,  $\sum_{w \in N_{G_2}(u)} x(w) = 0$ , and  $\mathbf{x}$  is also a Fiedler vector of  $G^*$ .

Denote by  $T(k, l, d)$  (see Fig. 1) a tree of order  $n$  obtained from a path  $P_d$  by attaching two stars  $K_{1,k}$  and  $K_{1,l}$  at its two end vertices of  $P_d$ , respectively, where  $k + l + d = n$ . In particular, if  $d = 1$ , then  $T(k, l, d) \cong K_{1,k+l}$ ; if  $k = 1$  and  $l = 1$ , then  $T(k, l, d) \cong P_n$ . Denote  $T_d := T(\lceil \frac{n-d}{2} \rceil, \lfloor \frac{n-d}{2} \rfloor, d)$ .

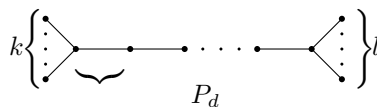


Fig. 1:  $T(k, l, d)$ .

LEMMA 2.3. [6] Among all trees of order  $n$  and diameter  $d + 1$ , the tree  $T_d$  is the unique graph with minimum algebraic connectivity.

LEMMA 2.4. If  $k, l \geq 2$ , then

$$(1) 0 \leq \alpha(T(k, l, d)) - \alpha(T(k - 1, l, d + 1)) \leq 1;$$

$$(2) \mu(T(k, l, d)) > \mu(T(k - 1, l, d + 1)) \text{ (see [8]).}$$

*Proof.* Note that  $\alpha(T(k, l, d)) = k + l + \lfloor \frac{d-1}{2} \rfloor = n - \lceil \frac{d+1}{2} \rceil$ ,  $\alpha(T(k - 1, l, d + 1)) = n - \lceil \frac{d+2}{2} \rceil$ , then  $\alpha(T(k, l, d)) = \alpha(T(k - 1, l, d + 1))$  when  $d$  is even;  $\alpha(T(k, l, d)) = \alpha(T(k - 1, l, d + 1)) + 1$  when  $d$  is odd. So, we have

$$0 \leq \alpha(T(k, l, d)) - \alpha(T(k - 1, l, d + 1)) \leq 1. \quad \square$$

LEMMA 2.5. If  $d_1 > d_2 \geq 2$ , then

$$\alpha(T_{d_1}) \leq \alpha(T_{d_2}) \text{ and } \mu(T_{d_1}) < \mu(T_{d_2}).$$

*Proof.* Since  $\alpha(T_{d_1}) = n - \lceil \frac{d_1+1}{2} \rceil$ ,  $\alpha(T_{d_2}) = n - \lceil \frac{d_2+1}{2} \rceil$ , we have  $\alpha(T_{d_1}) = \alpha(T_{d_2})$  if  $d_1 = d_2 + 1$  and  $d_2$  is even;  $\alpha(T_{d_1}) < \alpha(T_{d_2})$  otherwise.

Assume  $d_1 = d_2 + k$  ( $k \geq 1$ ). In the following, we will show that  $\mu(T_{d_1}) < \mu(T_{d_2})$  by induction on  $k$ . If  $k = 1$ , then by Lemmas 2.3 and 2.4, we have

$$\mu(T_{d_2}) > \mu\left(T\left(\left\lceil \frac{n-d_2}{2} \right\rceil - 1, \left\lfloor \frac{n-d_2}{2} \right\rfloor, d_2 + 1\right)\right) \geq \mu(T_{d_2+1}) = \mu(T_{d_1}).$$

So, we suppose that  $k \geq 2$  and  $\mu(T_{d_2+k-1}) < \mu(T_{d_2})$ . Then, by Lemmas 2.3 and 2.4, we have

$$\begin{aligned} \mu(T_{d_1}) &= \mu(T_{d_2+k}) \\ &\leq \mu\left(T\left(\left\lceil \frac{n-d_2-k+1}{2} \right\rceil - 1, \left\lfloor \frac{n-d_2-k+1}{2} \right\rfloor, d_2 + k\right)\right) \\ &< \mu(T_{d_2+k-1}) \\ &< \mu(T_{d_2}). \end{aligned}$$

Therefore, the proof of the lemma is complete.  $\square$

LEMMA 2.6. *If  $\alpha(T_d) = \alpha \geq 2$ , then  $2n - 2\alpha - 2 \leq d \leq 2n - 2\alpha - 1$  and, furthermore,  $\mu(T_d) \geq \mu(T_{2n-2\alpha-1})$  with equality if and only if  $d = 2n - 2\alpha - 1$ .*

*Proof.* First we note that  $\alpha(T_d) = n - \lceil \frac{d+1}{2} \rceil = \alpha \geq 2$ . Then we have  $\alpha \leq n - \frac{d+1}{2} \leq \alpha + \frac{1}{2}$ , that is,  $2n - 2\alpha - 2 \leq d \leq 2n - 2\alpha - 1$ .

If  $d = 2n - 2\alpha - 2$ , then by Lemmas 2.3 and 2.4,

$$\mu(T_d) > \mu(T(\lceil \frac{n-d}{2} \rceil - 1, \lfloor \frac{n-d}{2} \rfloor, d+1)) \geq \mu(T_{d+1}) = \mu(T_{2n-2\alpha-1}),$$

and thus,  $\mu(T_d) \geq \mu(T_{2n-2\alpha-1})$  with equality if and only if  $d = 2n - 2\alpha - 1$ . □

LEMMA 2.7. *Let  $G$  be a connected graph of order  $n$  with given stability number  $\alpha \geq \lceil \frac{n}{2} \rceil$ . Then  $G$  contains a spanning tree with stability number  $\alpha$ .*

*Proof.* If  $n \leq 3$ , then the result holds clearly. So we suppose  $n \geq 4$  and  $\alpha \geq 2$ . Let  $S = \{v_1, v_2, \dots, v_\alpha\}$  be a maximum stable set of  $G$  with  $|S| = \alpha$ , and let  $U = V(G) \setminus S$ .

Let  $H$  be the bipartite spanning subgraph of  $G$  with the bipartition  $\{S, U\}$ . Then  $\alpha(H) = \alpha$  as  $\alpha \geq \lceil \frac{n}{2} \rceil$ .

If  $H$  is connected, we can get a spanning tree  $T$  by the following algorithm:

- 1: set  $T := H$ ,  $i = 1$
- 2: while  $|N_T(v_{i+1}) \cap (\cup_{k=1}^i N_T(v_k))| \geq 2$  do, let  $u \in N_T(v_{i+1}) \cap (\cup_{k=1}^i N_T(v_k))$
- 3: delete the edges between  $v_{i+1}$  and  $(N_T(v_{i+1}) \cap (\cup_{k=1}^i N_T(v_k))) \setminus \{u\}$
- 4: replace  $T$  by  $T - E_i$ ,  $i$  by  $i + 1$  ( $E_i$  denote the set of edges deleted)
- 5: end while
- 6: return  $T$ .

Then  $T$  is a spanning tree of  $G$  with stability number  $\alpha(T) = \alpha(G)$ .

If  $H$  is not connected, let  $H_1, H_2, \dots, H_k$  ( $k \geq 2$ ) be the components of  $H$  with bipartitions  $(S_1, U_1), (S_2, U_2), \dots, (S_k, U_k)$ . Similar to the above discussion, each  $H_i$  contains a spanning tree  $T_i$  such that  $\alpha(T_i) = \alpha(H_i)$  for  $i = 1, 2, \dots, k$ . Since  $G$  is connected, there exists a spanning tree  $T$  of  $G$  obtained from  $T_1 \cup T_2 \cup \dots \cup T_k$  by adding  $k - 1$  edges between  $U_i$  and  $U_j$  ( $i, j = 1, 2, \dots, k$  and  $i \neq j$ ), and  $\alpha(T) = \alpha$ . □

LEMMA 2.8. [14] *Let  $G$  be a graph with  $PV(G) \neq \emptyset$ . Then there must exists a maximum stable set  $S$  of  $G$  such that  $PV(T) \subseteq S$  and  $PN(G) \cap S = \emptyset$ .*

**3. Main results.** In this section, we will characterize the graphs which have the minimum algebraic connectivity among all connected graphs with given stability number or covering number, respectively. Let  $\mathcal{T}_{n,\alpha}$  be the set of trees of order  $n$  with given stability number  $\alpha$ , and let

$$\mathcal{T}_{n,\alpha}^* = \{T : T \text{ is a tree of order } n \text{ with } \alpha(T) \geq \alpha\}.$$

THEOREM 3.1. *For a tree  $T \in \mathcal{T}_{n,\alpha}^*$ , where  $\alpha \geq \lceil \frac{n}{2} \rceil$ , we have*

$$\mu(T) \geq \mu(T_{d^*})$$

*with equality if and only if  $T \cong T_{d^*}$ , where  $d^* = 2n - 2\alpha - 1$ .*

*Proof.* Choose  $T \in \mathcal{T}_{n,\alpha}^*$  such that  $\mu(T)$  is as small as possible, where  $\alpha \geq \lceil \frac{n}{2} \rceil$ . First we note that if  $\alpha(T) = n - 1$ , then  $T \cong K_{1,n-1}$ ; and if  $\alpha(T) = \lceil \frac{n}{2} \rceil$ , then  $T \cong P_n$ . Thus, in either case, the result holds. So, in the following, we assume  $T \not\cong K_{1,n-1}$  and  $T \not\cong P_n$ . Let  $S$  be a maximum stable set of  $T$  with  $PV(T) \subseteq S$ , then by Lemma 2.8,  $PN(T) \cap S = \emptyset$ . We will show the following two claims.

Claim 1.  $T$  has exactly two pendant stars.

*Proof of Claim 1.* Suppose that  $T$  has more than two pendant stars. Let  $S_1, S_2$  be two pendant stars of  $T$  attached at  $u_1, u_2$ , respectively. Denote  $T_i = T - N(u_i) \cap PV(T)$ ,  $i = 1, 2$ . Then  $T = T_i(u_i) \diamond S_i(u_i)$ . Let  $\mathbf{x}$  be a Fiedler vector of  $T$ . We consider two cases.

Case 1.  $T$  is of Type I.

In this case,  $N_0 = \{v \in V(T) : x(v) = 0\} \neq \emptyset$ . By Lemma 2.1, Case B, we can let  $v_0$  be the characteristic vertex, and then the values of other vertices along any path in  $T$  starting in  $v_0$  are strictly increasing or strictly decreasing or zero.

If  $|N_0| = 1$ , then  $x(u_i) \neq 0$ ,  $i = 1, 2$ . Without loss of generality, we assume that  $|x(u_1)| \geq |x(u_2)| > 0$ . If  $x(u_1) \geq x(u_2) > 0$ , then we set

$$T' = T_2(u_1) \diamond S_2(u_2).$$

By Lemma 2.2, we have  $\mu(T') < \mu(T)$ . Note that  $S$  is also a stable set of  $T'$  as  $u_1 \notin S$ , then  $\alpha(T') \geq \alpha(T) \geq \alpha$ , and thus,  $T' \in \mathcal{T}_{n,\alpha}^*$ , which is a contradiction with the choice of  $T$ . If  $x(u_1) \leq x(u_2) < 0$ , let  $\mathbf{y}$  be a vector defined as

$$y(v_0) = x(v_0) = 0 \text{ and } y(v) = -x(v) \text{ for } v \in V(T) \setminus \{v_0\}.$$

It is easy to check that  $\mathbf{y}$  is also a Fiedler vector of  $T$  with  $y(u_1) \geq y(u_2) > 0$ . By an argument similar to the above, we can derive a contradiction.

If  $|N_0| \geq 2$ , then by Lemma 2.1,  $v_0$  has at least one neighbor not belong to  $N_0$ , say  $v_1 \in N(v_0) \setminus N_0$ . Then we can choose  $S_1$  such that  $v_1$  belongs to the unique  $(v_0, u_1)$ -path in  $T$ . By Lemma 2.1,  $x(u_1) \neq 0$ . If  $x(u_2) \neq 0$ , then by an argument similar to the above, we have a contradiction. If  $x(u_2) = 0$ , then  $|x(u_1)| > x(u_2)$ , and then we can have a contradiction similarly.

Case 2.  $T$  is of Type II.

In this case, by Lemma 2.1, Case A, there exists  $uv \in E(T)$  such that  $x(u) > 0$  and  $x(v) < 0$ , the values of vertices along any path in  $T$  which starts in  $u$  (or resp.,  $v$ ) and does not contain  $v$  (or resp.,  $u$ ) strictly increasing (or resp. decreasing).

Choose  $S_1$  (or resp.,  $S_2$ ) such that  $u_1$  (or resp.,  $u_2$ ) belongs to some path in  $T$  starting in  $u$  (or resp.,  $v$ ) and not containing  $v$  (or resp.,  $u$ ). Then

$$x(u_1) > 0 > x(u_2).$$

Similar to the proof of Case 1, we can get a contradiction.

By Claim 1, we have  $T \cong T(k, l, d)$  for some  $k, l, d$ . Without loss of generality, we assume that  $k \geq l$ . Recall that  $T \not\cong P_n$ . Thus,  $k \geq 2$ .

Claim 2.  $\alpha(T) = \alpha$ .

*Proof of Claim 2.* Suppose that  $\alpha(T) > \alpha$ . Then by Lemma 2.4, we have  $\alpha(T(k-1, l, d+1)) \geq \alpha(T) - 1 \geq \alpha$  and  $\mu(T) > \mu(T(k-1, l, d+1))$ . Note that  $T(k-1, l, d+1) \in \mathcal{T}_{n,\alpha}^*$ , and hence which is a contradiction with the choice of  $T$ .

By Claims 1 and 2, we get  $T \cong T_d$  and  $\alpha(T_d) = \alpha(T) = \alpha$ . By Lemma 2.6, the result holds immediately.  $\square$

**COROLLARY 3.2.** *Among all trees of order  $n$  with stability number  $\alpha \geq \lceil \frac{n}{2} \rceil$ ,  $T_{d^*}$  is the unique graph with minimum algebraic connectivity, where  $d^* = 2n - 2\alpha - 1$ .*

*Proof.* Note that  $\alpha(T_{d^*}) = \alpha$  and  $\mathcal{T}_{n,\alpha} \subseteq \mathcal{T}_{n,\alpha}^*$ . By Theorem 3.1, we get the result immediately.  $\square$

**THEOREM 3.3.** *Among all connected graphs of order  $n$  with stability number  $\alpha \geq \lceil \frac{n}{2} \rceil$ ,  $T_{2n-2\alpha-1}$  is the unique minimizing graph.*

*Proof.* If  $\alpha = n - 1$ , then the result holds as  $T_1 \cong K_{1,n-1}$  and  $K_{1,n-1}$  is the unique graph with stability number  $n - 1$ . So we suppose  $\lceil \frac{n}{2} \rceil \leq \alpha < n - 1$ . Let  $G$  be a minimizing graph. By Lemma 2.7,  $G$  contains a spanning tree  $T$  with stability number at least  $\alpha$ . By Theorem 3.1,  $\mu(G) \geq \mu(T) \geq \mu(T_{2n-2\alpha-1})$ . Since  $G$  is minimizing, we have  $\mu(G) = \mu(T) = \mu(T_{2n-2\alpha-1})$ . Furthermore,  $T \cong T_{2n-2\alpha-1}$  by Theorem 3.1. Note that  $\alpha(T_{2n-2\alpha-1}) = \alpha = \alpha(G)$ .

In the following, we will show that  $E(G) \setminus E(T_{2n-2\alpha-1}) = \emptyset$ , i.e.,  $G \cong T_{2n-2\alpha-1}$ . Suppose  $E(G) \setminus E(T_{2n-2\alpha-1}) \neq \emptyset$ . Let  $\mathbf{x}$  be a unit Fiedler vector of  $G$ , then

$$\begin{aligned} \mu(G) &= \mathbf{x}^T L(G) \mathbf{x} = \sum_{uv \in E(G)} [x(u) - x(v)]^2 \\ &= \sum_{uv \in E(G) \setminus E(T_{2n-2\alpha-1})} [x(u) - x(v)]^2 + \sum_{uv \in E(T_{2n-2\alpha-1})} [x(u) - x(v)]^2 \\ &\geq \sum_{uv \in E(T_{2n-2\alpha-1})} [x(u) - x(v)]^2 \geq \mu(T_{2n-2\alpha-1}). \end{aligned}$$

Note that  $\mu(G) = \mu(T_{2n-2\alpha-1})$ , and hence,  $\mathbf{x}$  is also a Fiedler vector of  $T_{2n-2\alpha-1}$  and  $x(u) = x(v)$  for each  $uv \in E(G) \setminus E(T_{2n-2\alpha-1})$ . By Lemma 2.1,  $E(G) \setminus E(T_{2n-2\alpha-1})$  consists of edges joining the pendant vertices of the same pendant star, and thus,  $\alpha(T_{2n-2\alpha-1} + uv) < \alpha(T_{2n-2\alpha-1})$  for each  $uv \in E(G) \setminus E(T_{2n-2\alpha-1})$ . Hence,  $\alpha(G) < \alpha(T_{2n-2\alpha-1})$ , a contradiction. Therefore, we have  $E(G) \setminus E(T_{2n-2\alpha-1}) = \emptyset$ ,  $G \cong T_{2n-2\alpha-1}$ .  $\square$

**THEOREM 3.4.** *Among all connected graphs of order  $n$  with covering number  $2 \leq \beta \leq \lfloor \frac{n}{2} \rfloor$ ,  $T_{2\beta-1}$  is the unique minimizing graph.*

*Proof.* By Theorem 3.2, the result holds from the fact that for any graph  $G$ ,  $\alpha(G) + \beta(G) = n$ .  $\square$

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