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ON THE GROUP INVERTIBILITY OF OPERATORS∗

CHUN YUAN DENG†

Abstract. The main topic of this paper is the group invertibility of operators in Hilbert spaces. Conditions for the existence of the group inverses of products of two operators and the group invertibility of anti-triangular block operator matrices are studied. The equivalent conditions related to the reverse order law for the group inverses of operators are obtained.

Key words. Group inverse, Block operator matrix, EP operator.

AMS subject classifications. 15A09, 47A05.

1. Introduction. Let \( \mathcal{H} \) be an infinite-dimensional complex Hilbert space. Denote by \( \mathcal{B}(\mathcal{H}) \) the Banach algebra of all bounded linear operators on \( \mathcal{H} \); and the range and nullspace of \( T \) by \( \mathcal{R}(T) \) and \( \mathcal{N}(T) \), respectively. Recall that the adjoint \( T^* \) of a linear operator \( T \) on a Hilbert space \( \mathcal{H} \) is defined as the linear operator satisfying the condition \( (Tx, y) = (x, T^*y) \) for all \( x, y \in \mathcal{H} \). When \( T = T^* \), \( T \) is called self-adjoint. Let \( P_U \) be the orthogonal projection onto closed subspace \( U \subset \mathcal{H} \) and \( P_{U,V} \) be the idempotent with \( \mathcal{R}(P_{U,V}) = U \) and \( \mathcal{N}(P_{U,V}) = V \). The direct sum and the orthogonal direct sum are denoted by \( U \oplus V \) and \( U \oplus \perp V \), respectively. It is clear \( \mathcal{R}(P_U) + \mathcal{N}(P_U) = U \oplus \perp U = \mathcal{H} \) and \( \mathcal{R}(P_{U,V}) + \mathcal{N}(P_{U,V}) = U \oplus V = \mathcal{H} \). An operator \( T \) is generalized invertible if there is an operator \( S \) such that \((I)\) \( TST = T \). The operator \( S \) is not unique in general. In order to force its uniqueness, further conditions have to be imposed. The most convenient additional conditions are

\[
\begin{align*}
\text{(II)} & \quad STS = S, \quad \text{(III)} \quad (TS)^* = TS, \quad \text{(IV)} \quad (ST)^* = ST, \quad \text{(V)} \quad TS = ST.
\end{align*}
\]

One also considers the condition \((I_k)\) \( T^kST = T^k \), where \( k \) is a fixed positive integer. Clearly, \((I) = (I_1)\). Elements \( S \in \mathcal{B}(\mathcal{H}) \) satisfying \((I)\) are called \{1\}-inverses of \( T \) and is denoted by \( S = T^{-} \). Similarly, \((I, II, V)\)-inverse is called group inverse and is denoted by \( S = T^{-} \). Similarly, \((I, II, III, IV)\)-inverse is Moore-Penrose inverse and is denoted by \( S = T^{+} \). And \((I_k, II, V)\)-inverse is Drazin inverse and is denoted by \( S = T^{D} \), where \( k = \text{ind}(T) \), the Drazin index of \( T \), is the smallest nonnegative integer for which \( \mathcal{R}(T^{k+1}) = \mathcal{R}(T^k) \) and \( \mathcal{N}(T^{k+1}) = \mathcal{N}(T^k) \) (see [17]). In particular, when

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1 = \text{ind}(T)$, the operator $T^D$ is called the group inverse of $T$, and is denoted by $T^\#$. When $\text{ind}(T) = 0$, the group inverse is the standard inverse, i.e. $T^\# = T^{-1}$. An operator $T \in \mathcal{B}(\mathcal{H})$ has the Moore-Penrose inverse $T^+$ if and only if $\mathcal{R}(T)$ is closed. The Moore-Penrose inverse $T^+$ is unique and $TT^+ = P_{\mathcal{R}(T)}$ and $T^+T = P_{\mathcal{R}(T^*)}$ (see [5]). In addition, if $T \in \mathcal{B}(\mathcal{H})$ is positive, then $TT^+ = (TT^+)^* = (T^+)^*T^+ = T^+T$.

An element $T \in \mathcal{B}(\mathcal{H})$ is said to be EP (see [12]) if $T^\# = T^+$. And $T$ is EP if and only if $T^+T = TT^+$ if and only if $T$ is group invertible and $T^\# T$ is self-adjoint.

In this paper, we often restrict our attention to operator with $\text{ind}(T) \leq 1$; i.e., the group invertible operators. For $T \in \mathcal{B}(\mathcal{H})$, the group inverse (if exists) of $T$ is unique (see [5, 10]). An element $T \in \mathcal{B}(\mathcal{H})$ is group invertible if and only if there is an idempotent $P \in \mathcal{B}(\mathcal{H})$ such that $T + P$ is invertible, $TP = 0$ and $TP = PT$. If these conditions are satisfied, the group inverse $T^\#$ of $T$ is given by $T^\# = (T + P)^{-1}(I - P)$, and the idempotent $P = T^\varepsilon = I - TT^\#$ (see [16]). In this paper, we give necessary and sufficient conditions for a product of two operators to be group invertible. Some equivalent conditions by the reverse order law of group inverses are obtained. Furthermore, using the standard inverses of operators, we give some equivalent conditions for the existence of the group inverses of anti-triangular matrices.

2. The group invertibility of the products of two operators. First, we recall that, for a bounded linear operator $T$ and the idempotent $P_{\mathcal{L}, \mathcal{M}}$, the following properties hold, which was given in [18] for Hilbert C*-modules.

**Lemma 2.1.** (See [18]) Let $\mathcal{L}$ and $\mathcal{M}$ be closed subspaces of $\mathcal{H}$, let also $P_{\mathcal{L}, \mathcal{M}}$ be an idempotent on $\mathcal{L}$ along $\mathcal{M}$.

(i) $P_{\mathcal{L}, \mathcal{M}}T = T$ if and only if $\mathcal{R}(T) \subset \mathcal{L}$.

(ii) $TP_{\mathcal{L}, \mathcal{M}} = T$ if and only if $\mathcal{N}(T) \supset \mathcal{M}$.

For $A, B \in \mathcal{B}(\mathcal{H})$, we say $A$ and $B$ are similar (written $A \sim B$) if there is an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $SA = BS$. We remark that any operator $T \in \mathcal{B}(\mathcal{H})$ has a matrix representation on $\mathcal{H} = \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T^*)$ of the form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & 0 \end{pmatrix}, \quad T_{11} = T|_{\mathcal{R}(T)} \in B(\mathcal{R}(T)), \quad T_{12} \in B(\mathcal{N}(T^*), \mathcal{R}(T)).$$

The following properties of group inverses will be used later.

**Lemma 2.2.** Let $T, B \in \mathcal{B}(\mathcal{H})$.

(i) If $T \sim B$, then $B$ is group invertible if and only if $T$ is group invertible.

(ii) If $T$ is group invertible, then $\mathcal{R}(T)$ is closed; $(T^*)^\# = (T^\#)^*$; $(T^k)^\# = (T^\#)^k$ for each nonnegative integer $k$; $T^\varepsilon = P_{\mathcal{N}(T), \mathcal{R}(T)}$ and $TT^\# = P_{\mathcal{R}(T), \mathcal{N}(T)}$. 


(iii) The following are equivalent:

(a) \( T \) is group invertible.

(b) \( \mathcal{R}(T) = \mathcal{R}(T^k), \mathcal{N}(T) = \mathcal{N}(T^k), k \geq 2 \).

(c) \( T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & 0 \end{pmatrix} \) with respect to the space decomposition \( \mathcal{H} = \mathcal{R}(T) \oplus \mathcal{R}(T)^{\perp} \) \( \mathcal{R}(T)^{\perp} \), where \( T_{11} \) is invertible.

(d) \( T = T_0 \oplus 0 \), with respect to the space decomposition \( \mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T) \), where \( T_0 \) is invertible.

If \( A \) and \( B \) are \( n \times n \) complex matrices, then \( (AB)^D = A[(BA)^D]^2B \) (see [5]). For \( A, B \in \mathcal{B}(\mathcal{H}) \), if \( BA \) is group invertible, then it is easy to get the following results.

**Lemma 2.3.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \). If \( BA \) is group invertible, then \( AB \) is Drazin invertible with \( \text{ind}(AB) \leq 2 \) and \( (AB)^D = A \begin{pmatrix} (BA)^\# \end{pmatrix}^2 B \). If both \( AB \) and \( BA \) are group invertible then

\[
(2.1) \quad (AB)^\# = A \begin{pmatrix} (BA)^\# \end{pmatrix}^2 B, \quad (AB)^\# A = A(AB)^\#, \quad \text{and} \quad B(AB)^\# = (BA)^\# B.
\]

**Proof.** Let \( X = A \begin{pmatrix} (BA)^\# \end{pmatrix}^2 B \). Clearly, \( XABX = X \) and \( ABX = XAB = A(AB)^\# B \). From \( (AB)^3X = (AB)^2A(AB)^\# B = A(AB)^2(AB)^\# B = (AB)^2 \), we get that \( X \) is \((I, II, V)\)-inverse of \( AB \). Hence, \( (AB)^D = X \) and \( \text{ind}(AB) \leq 2 \). Moreover, if \( AB \) and \( BA \) are group invertible, then \( (AB)^\# = (AB)^D = A \begin{pmatrix} (BA)^\# \end{pmatrix}^2 B, (AB)^\# A = A \begin{pmatrix} (BA)^\# \end{pmatrix}^2 BA = A(AB)^\# \) and \( B(AB)^\# = BA \begin{pmatrix} (BA)^\# \end{pmatrix}^2 B = (BA)^\# B \). \( \square \)

For a diagonal operator \( M = A \oplus D \), we know \( \mathcal{R}(M) = \mathcal{R}(A) \oplus \mathcal{R}(D), \mathcal{N}(M) = \mathcal{N}(A) \oplus \mathcal{N}(D) \) and \( M \) is group invertible if and only if \( A \) and \( D \) are group invertible. For an upper triangular finite matrix \( \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \), R.E. Hartwig and J.M. Shao had proved that \( \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^\# \) exists if and only if \( A^\# \) and \( D^\# \) exist and \( (I - AA^\#)B(I - DD^\#) = 0 \) (see [11] Theorem 1). In this case, \( \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^\# = \begin{pmatrix} A^\# & Y \\ 0 & D^\# \end{pmatrix} \), where \( Y = (A^\#)^2BD^\pi + A^\pi B(D^\#)^2 - A^\# BD^\# \). The group inverse over a Bezout domain had been studied by C. Cao and analogous results had been obtained (see [7] Theorem...
As for bounded linear operators, it should be noted, however, if \((A \quad B)
D\) is group invertible, we cannot conclude that both \(A\) and \(D\) are group invertible. This can be seen from the following example.

**Example 2.4.** Define \(M\) on \(l^2 \oplus l^2\), with \(l^2\) being the Hilbert space of square summable sequences, by
\[
M = \begin{pmatrix}
U & I - UU^* \\
0 & U^*
\end{pmatrix},
\]
where \(U\) is the unilateral shift operator defined by
\[
U(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots).
\]
Then \(M\) is group invertible (in fact \(M\) invertible). But not both \(U\) and \(U^*\) are group invertible. In fact, \(U\) is an isometry. The range of \(U\) is not \(l^2\) but a proper subspace of \(l^2\), the subspace of vectors with vanishing first coordinate. Since the spectrum of the unilateral shift \(U\) is the closed unit disc, 0 is not the isolated point of the spectrum of \(U\). Hence, neither \(U\) nor \(U^*\) is group invertible.

Recall that \(\text{asc}(T)\) (resp., \(\text{desc}(T)\)), the ascent (resp., descent) of \(T \in \mathcal{B}(\mathcal{H})\), is the smallest non-negative integer \(n\) such that \(\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\) (resp., \(\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\)). It is well known that \(\text{desc}(T) = \text{asc}(T)\) if \(\text{asc}(T)\) and \(\text{desc}(T)\) are finite \([17]\) and this common value is known as the Drazin index of \(T\), denoted by \(\text{ind}(T)\). In \([6, 7]\), some necessary and sufficient conditions for the existence of the group inverse for square matrix are given. Here, we will show that some results do not hold for operators in Hilbert spaces. As for the group inverse of upper triangular operator matrices, we have the following result.

**Theorem 2.5.** Let \(\mathcal{H}, \mathcal{K}\) be Hilbert spaces, \(M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\) be an operator on \(\mathcal{H} \oplus \mathcal{K}\). The following assertions hold:

(i) Assume that \(D^\#\) exists (resp., \(A^\#\) exists). Then \(M^\#\) exists if and only if \(A^\#\) exists (resp., \(D^\#\) exists) and \(A^\#BD^\# = 0\).

(ii) Assume that \(A^\#\) and \(D^\#\) exist. Then \(M^\#\) exists if and only if \(A^\#BD^\# = 0\). In this case,
\[
\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^\# = \begin{pmatrix} A^\# & Y \\ 0 & D^\# \end{pmatrix}, \quad \text{where } Y = (A^\#)^2BD^\# + A^\#B(D^\#)^2 - A^\#BD^\#.
\]

(iii) Assume that \(\mathcal{H}\) (resp., \(\mathcal{K}\)) is finite dimensional. Then \(M^\#\) exists if and only if \(A^\#, D^\#\) exist and \(A^\#BD^\# = 0\).
Proof. (i) Necessity: Assume that $M^\#$ exists and note that

$$M^2 = \begin{pmatrix} A^2 & AB + BD \\ 0 & D^2 \end{pmatrix} \quad \text{and} \quad M^3 = \begin{pmatrix} A^3 & A^2B + ABD + BD^2 \\ 0 & D^3 \end{pmatrix}.$$  

Since $M$ and $D$ are group invertible, $\mathcal{R}(M) = \mathcal{R}(M^k)$, $\mathcal{N}(M) = \mathcal{N}(M^k)$, $\mathcal{R}(D) = \mathcal{R}(D^k)$ and $\mathcal{N}(D) = \mathcal{N}(D^k)$ for an integer $k \geq 2$. Thus, for every $x \in \mathcal{N}(A^2)$, $x \oplus 0 \in \mathcal{N}(M^2) = \mathcal{N}(M)$. Hence, $Ax = 0$ and $\mathcal{N}(A^2) \subset \mathcal{N}(A)$. Since $\mathcal{N}(A^2) \subset \mathcal{N}(A)$ is trivial, we get $\mathcal{N}(A^2) = \mathcal{N}(A)$.

For every $z \in \mathcal{R}(A)$, there exists $x \in \mathcal{H}$ such that

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix} \in \mathcal{R}(M) = \mathcal{R}(M^3).$$

Hence, there exists $u \in \mathcal{H}$ and $v \in \mathcal{K}$ such that

$$\begin{pmatrix} A^3 & A^2B + ABD + BD^2 \\ 0 & D^3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}.$$  

We deduce that $v \in \mathcal{N}(D^3) = \mathcal{N}(D)$. Therefore, $z = A^3u + A^2Bv = A^2(Au + Bv) \in \mathcal{R}(A^2)$. Hence $\mathcal{R}(A) \subset \mathcal{R}(A^2)$. Since $\mathcal{R}(A^2) \subset \mathcal{R}(A)$ is trivial, we get $\mathcal{R}(A^2) = \mathcal{R}(A)$.

Hence, $\text{ind}(A) = 1$, i.e., $A$ is group invertible. Now, $A$ as an operator acting on $\mathcal{N}(A^*) \oplus \mathcal{R}(A^*)$, $D$ as an operator acting on $\mathcal{N}(D^*) \oplus \mathcal{R}(D^*)$ and $B$ as an operator from $\mathcal{N}(D^*) \oplus \mathcal{R}(D^*)$ into $\mathcal{N}(A^*) \oplus \mathcal{R}(A^*)$ have the forms

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_3 \\ B_4 & B_2 \end{pmatrix},$$

respectively, where $A_1$ and $D_1$ are invertible. Let

$$S = \begin{pmatrix} I & 0 & -B_1D_1^{-1} \\ 0 & I & 0 \\ 0 & -B_4D_1^{-1} & 0 \end{pmatrix}.$$  

Then $S$ is invertible,

$$S^{-1} = \begin{pmatrix} I & B_1D_1^{-1} & 0 \\ 0 & B_4D_1^{-1} & I \\ 0 & I & 0 \end{pmatrix},$$

and

$$S MS^{-1} = \begin{pmatrix} A_1 & A_1B_1D_1^{-1} & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
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Since $M$ is group invertible, $N = \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix}$ exists. Obviously $N^2 = 0$. So $N = N^2 N^\# = 0$. Hence, $B_2 = 0$; i.e., $A^* B D^* = 0$.

Sufficiency: Assume that both $A^\#$ and $D^\#$ exist. By \((2.2)\) and \((2.3)\), $A^* B D^* = 0$, and thus,

$$\text{SMS}^{-1} = \begin{pmatrix} A_1 & A_1 B_1 D_1^{-1} \\ 0 & D_1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is group invertible. Lemma \(2.2(i)\) implies that $M$ is group invertible.

(ii) The claim follows easily from \((2.3)\), and the formula for the group inverse can be readily verified.

(iii) The sufficiency has been shown in (i). For the proof of the necessity, since the dimension of $H$ is finite, $\text{asc}(A)$ and $\text{desc}(A)$ are finite and $\text{desc}(A) = \text{asc}(A)$ (see \([17]\)).

If $M$ is group inverse, by the proof of (i), $\mathcal{N}(A^2) = \mathcal{N}(A)$. So $\text{desc}(A) = \text{asc}(A) \leq 1$ and $A$ is group invertible. The necessity follows immediately by (i) and the fact that $M$ and $A$ are group invertible. \(\Box\)

In \([7]\), C. Cao and J. Li shown that $BA$ is group invertible if $AB$ is group invertible and $AB \sim BA$, where $A, B \in R^{n \times n}$ and $R$ is a Bezout domain. We can generalize this result to the operators on an arbitrary Hilbert space.

**Theorem 2.6.** Let $A, B \in \mathcal{B}(H)$. If any two of the following hold, then the remaining one holds:

(i) $(AB)^\#$ exists; (ii) $(BA)^\#$ exists; (iii) $AB \sim BA$.

**Proof.** (i),(ii)$\Rightarrow$(iii): Let $AB$ and $BA$ be group invertible, $P = (AB)^* = I - AB(AB)^*$ and $Q = (BA)^* = I - BA(BA)^*$. Then $AB$ as an operator acting on $\mathcal{N}(P) \oplus \mathcal{R}(P)$, $BA$ as an operator acting on $\mathcal{N}(Q) \oplus \mathcal{R}(Q)$, $A$ as an operator from $\mathcal{N}(Q) \oplus \mathcal{R}(Q)$ into $\mathcal{N}(P) \oplus \mathcal{R}(P)$ and $B$ as an operator from $\mathcal{N}(P) \oplus \mathcal{R}(P)$ into $\mathcal{N}(Q) \oplus \mathcal{R}(Q)$ can be written as

$$AB = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_3 \\ B_4 & B_2 \end{pmatrix},$$

respectively, where $X \in \mathcal{B}(\mathcal{N}(P))$ and $Y \in \mathcal{B}(\mathcal{N}(Q))$ are invertible. Since $Q = I - BA(BA)^* = I - B(AB)^* A$ (by Lemma \(2.3\), $AQ = A - AB(AB)^* A = PA$; i.e.,

$$\begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & A_3 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A_4 & A_2 \end{pmatrix}$$

Hence, $A_i = 0$, $i = 3, 4$ and $A = A_1 \oplus A_2$. Similarly, $QB = BP$, which implies that $B_i = 0$, $i = 3, 4$ and $B = B_1 \oplus B_2$. Thus, $X = A_1 B_1$ and $Y = B_1 A_1$ are invertible,
$A_2B_2 = 0$ and $B_2A_2 = 0$. From $X = A_1B_1$ and $Y = B_1A_1$, we conclude that
\[
\mathcal{N}(Q) = \mathcal{R}(BA) \subset \mathcal{R}(B_1A_1) \subset \mathcal{R}(B_1) \subset \mathcal{N}(Q) \quad \text{and} \quad \mathcal{N}(B_1) \subset \mathcal{N}(A_1B_1) = \{0\}.
\]
Thus, $B_1$ is invertible. Let $S = B_1 \oplus I$. Then $SAB = BAS$, i.e., $AB \sim BA$. The implications $(i),(iii) \Rightarrow (ii)$ and $(ii),(iii) \Rightarrow (i)$ are obvious by Lemma 2.2(i). \(\square\)

**Theorem 2.7.** If $A, B \in \mathcal{B}(\mathcal{H})$ are group invertible and $\mathcal{R}(A) = \mathcal{R}(B)$, then $AB$ and $BA$ are group invertible. Furthermore,
\[
(AB)^\# = B^\# A^\# B^\# B \quad \text{and} \quad (BA)^\# = A^\# B^\# A^\# A.
\]

**Proof.** Since $A, B \in \mathcal{B}(\mathcal{H})$ are group invertible and $\mathcal{R}(A) = \mathcal{R}(B)$, we have $BB^\# A = A$ and $AA^\# B = B$ by Lemma 2.1(i). Let $X = B^\# A^\# B^\# B$. Then
\[
XAB = B^\# A^\# B^\# BAB = B^\# A^\# AB = B^\# B, \quad XABX = B^\# BX = X,
\]
\[
ABX = ABB^\# A^\# B^\# B = AA^\# B^\# B = B^\# B, \quad ABXAB = B^\# BAB = AB,
\]
i.e., $X = (AB)^\#$. Analogously, $(BA)^\# = A^\# B^\# A^\# A$. \(\square\)

If $A, B \in \mathcal{B}(\mathcal{H})$ are EP operators and $\mathcal{R}(A) = \mathcal{R}(B)$, then $\mathcal{R}(A^*) = \mathcal{R}(A) = \mathcal{R}(B) = \mathcal{R}(B^*)$, hence $A^\# B^\# B = A^\#$ and $B^\# A^\# A = B^\#$ by Lemma 2.1(ii).

**Corollary 2.8.** (See [9, Theorem 5]) If $A, B \in \mathcal{B}(\mathcal{H})$ are EP operators and $\mathcal{R}(A) = \mathcal{R}(B)$, then $AB$ and $BA$ are EP operators. Furthermore,
\[
(AB)^\# = (AB)^+ = B^+ A^+ = B^\# A^\# \quad \text{and} \quad (BA)^\# = (BA)^+ = A^+ B^+ = A^\# B^\#.
\]

Theorem 2.7 also implies that $A^2$ is group invertible and $(A^2)^\# = (A^\#)^2$ if $A \in \mathcal{B}(\mathcal{H})$ is group invertible.

**Theorem 2.9.** Let $A, B \in \mathcal{B}(\mathcal{H})$. Then $AB$ and $BA$ are group invertible if and only if $\mathcal{R}(AB)$, $\mathcal{R}(BA)$ are closed, and
\[
(i) \ \mathcal{R}(AB) = \mathcal{R}(ABA), \quad (ii) \ \mathcal{R}(A^* B^*) = \mathcal{R}(A^* B^* A^*),
\]
\[
(iii) \ \mathcal{R}(BA) = \mathcal{R}(BAB), \quad (iv) \ \mathcal{R}(B^* A^*) = \mathcal{R}(B^* A^* B^*).
\]

**Proof.** Necessity: If $AB$ and $BA$ are group invertible, then $\mathcal{R}(AB)$ and $\mathcal{R}(BA)$ are closed (by Lemma 2.2(ii)). The group invertibility of $AB$ implies that
\[
\mathcal{R}(AB) = \mathcal{R}((AB)^2) \subset \mathcal{R}(ABA) \subset \mathcal{R}(AB).
\]
So \( \mathcal{R}(AB) = \mathcal{R}(ABA) \). Analogously, since \( BA, A^*B^* \) and \( B^*A^* \) are group invertible, the remaining equations in (2.4) can be proved in the same way.

Sufficiency: If \( \mathcal{R}(AB), \mathcal{R}(BA) \) are closed and (2.3) holds, then

\[ \mathcal{R}(AB) = \mathcal{R}(ABA) = AR(BA) = A\mathcal{R}(BAB) = \mathcal{R}(ABAB) = \mathcal{R}((AB)^2). \]

Since \( \mathcal{R}(A^*B^*), \mathcal{R}(B^*A^*) \) are also closed, \( \mathcal{R}(B^*A^*) = \mathcal{R}(B^*A^*B^*) = B^*\mathcal{R}(A^*B^*) = B^*\mathcal{R}(A^*B^*) = B^*\mathcal{R}(A^*B^*) = \mathcal{R}((B^*A^*)^2). \) Thus \( N(AB) = \mathcal{R}(B^*A^*) = \mathcal{R}((B^*A^*)^{1/2}) = N((AB)^2). \) This implies \( \text{ind}(AB) \leq 1 \). Hence, \( AB \) is group invertible.

Analogously, \( BA \) is group invertible. \( \Box \)

**Corollary 2.10.** Let \( A, B \in B(H) \).

(i) If \( \mathcal{R}(A) \) is closed, \( \mathcal{R}(A) = \mathcal{R}(ABA) \) and \( \mathcal{R}(A^*) = \mathcal{R}(A^*B^*A^*), \) then \( AB \) and \( BA \) are group invertible.

(ii) If \( \mathcal{R}(A) \) and \( \mathcal{R}(B) \) are closed, \( \mathcal{R}(A) = \mathcal{R}(AB), \mathcal{R}(B) = \mathcal{R}(BA), \mathcal{R}(A^*) = \mathcal{R}(A^*B^*) \) and \( \mathcal{R}(B^*) = \mathcal{R}(B^*A^*), \) then \( AB \) and \( BA \) are group invertible.

(iii) If \( \mathcal{R}(BA) \) is closed, \( AB \) is group invertible, \( \mathcal{R}(A) = \mathcal{R}(AB) \) and \( \mathcal{R}(B^*) = \mathcal{R}(B^*A^*), \) then \( BA \) is group invertible.

**Proof.** (i) Since \( \mathcal{R}(A) \) is closed, \( \mathcal{R}(A) = \mathcal{R}(ABA) \subseteq \mathcal{R}(AB) \subseteq \mathcal{R}(A) \) and \( \mathcal{R}(A^*) = \mathcal{R}(A^*B^*A^*) \subseteq \mathcal{R}(A^*B^*) \subseteq \mathcal{R}(A^*), \) we get \( \mathcal{R}(AB), \mathcal{R}(BA) \) are closed and (i), (ii) in (2.3) hold. Similarly, since \( \mathcal{R}(BAB) = BR(ABA) = BR(A) = \mathcal{R}(BA) \) and \( \mathcal{R}(B^*A^*B^*A^*) = B^*\mathcal{R}(A^*B^*A^*) = B^*\mathcal{R}(A^*) = \mathcal{R}(B^*A^*), \) we get \( \mathcal{R}(BA) = \mathcal{R}(BAB) \subseteq \mathcal{R}(BA) \) and \( \mathcal{R}(B^*A^*) = \mathcal{R}(B^*A^*B^*A^*) \subseteq \mathcal{R}(B^*A^*B^*) \subseteq \mathcal{R}(B^*A^*). \) Hence, (iii) and (iv) in (2.3) hold. By Theorem 2.9, \( AB \) and \( BA \) are group invertible.

(ii) Note that \( \mathcal{R}(AB) = AR(B) = AR(BA) = \mathcal{R}(ABA) \). Analogously, we can show (ii)–(iv) in (2.3) hold, which gives the assertion via Theorem 2.9.

(iii) If \( (AB)^\# \) exists, \( \mathcal{R}(AB) = \mathcal{R}(ABA) \) by (2.3). Analogously, \( \mathcal{R}(B^*A^*) = \mathcal{R}(B^*A^*B^*A^*). \) Moreover, \( \mathcal{R}(BA) = BR(A) = BR(AB) = \mathcal{R}(BAB) \) and \( \mathcal{R}(A^*B^*) = A^*\mathcal{R}(B^*) = A^*\mathcal{R}(B^*A^*A^*) = \mathcal{R}(A^*B^*A^*). \) These give the assertion via Theorem 2.9. \( \Box \)

Next, the equivalent conditions related to the reverse order law for the group inverses of operators are studied.

**Theorem 2.11.** Let \( A, B, AB \in B(H) \) be group invertible. Then

\( (AB)^\# = B^#A^\# \) if and only if \( (I - A^\#)BA^\# = 0 \), \( B^#(I - A^\#) = (AB)^\# A. \)

In addition, if \( A, B, BA^\# \) are group invertible, then the following are equivalent:

(i) \( (AB)^\# = B^#A^\#; \)
\( (i) (BA)^\# = A^\# B^\#; \)

\( (ii) A = A_1 \oplus 0, B = B_1 \oplus B_2 \) and \( B_1^\# = (A_1B_1)^\# A_1 \) with respect to the space decomposition \( H = \mathcal{N}(A^\#) \oplus \mathcal{R}(A^\#) \), where \( A_1 \) is invertible;

\( (iv) A = A_1 \oplus 0, B = B_1 \oplus B_2 \) and \( B_1^\# = A_1(A_1B_1)^\# \) with respect to the space decomposition \( H = \mathcal{N}(A^\#) \oplus \mathcal{R}(A^\#) \), where \( A_1 \) is invertible.

**Proof.** Since \( A \) and \( B \) are group invertible, \( A, A^\#, B \) and \( B^\# \) as bounded linear operators acting on \( \mathcal{N}(A^\#) \oplus \mathcal{R}(A^\#) \) can be written as

\[
(2.6) A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^\# = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_3 \\ B_4 & B_2 \end{pmatrix}, \quad B^\# = \begin{pmatrix} C_1 & C_3 \\ C_4 & C_2 \end{pmatrix},
\]

respectively. Since \( AB = \begin{pmatrix} A_1B_1 & A_1B_3 \\ 0 & 0 \end{pmatrix} \) is group invertible, by Theorem 2.5(i), we get

\[
[I - A_1B_1(A_1B_1)^\#]A_1B_3 = 0 \quad \text{and} \quad (AB)^\# = \begin{pmatrix} (A_1B_1)^\# & [(A_1B_1)^\#]^2 A_1B_3 \\ 0 & 0 \end{pmatrix}.
\]

From \( (AB)^\# = B^\# A^\# \) we get

\[
\begin{pmatrix} (A_1B_1)^\# & [(A_1B_1)^\#]^2 A_1B_3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}.
\]

It follows that \( C_4 = 0, C_1 = (A_1B_1)^\# A_1 \) and \( [(A_1B_1)^\#]^2 A_1B_3 = 0 \). So

\[
B_3 = A_1^{-1}A_1B_3 = A_1^{-1}[A_1B_1(A_1B_1)^\# A_1B_3] = A_1^{-1}[A_1B_1]^2 [(A_1B_1)^\#]^2 A_1B_3 = 0.
\]

Note that \( A^\# = 0 \oplus I \). We get

\[
(I - A^\#)BA^\# = \begin{pmatrix} 0 & B_3 \\ 0 & 0 \end{pmatrix} = 0, \quad B^\#(I - A^\#) = \begin{pmatrix} (A_1B_1)^\#A_1 & 0 \\ 0 & 0 \end{pmatrix} = (AB)^\# A.
\]

On the other hand, if \( (I - A^\#)BA^\# = 0 \), then \( B_3 = 0 \) and \( (AB)^\# = (A_1B_1)^\# \oplus 0 \) by (2.6). If \( B^\#(I - A^\#) = (AB)^\# A \), by (2.6) again, \( C_1 = (A_1B_1)^\# A_1 \) and \( C_4 = 0 \). Hence, \( (AB)^\# = B^\# A^\# \).

Now, assume that \( A, B, AB, BA^\# \) are group invertible.

\( (i) \Rightarrow (iii) \): It is clear that \( (iii) \Rightarrow (i) \). So we only need to show that \( (i) \Rightarrow (iii) \). Note that \( (AB)^\# = B^\# A^\# \) if and only if \( A, A^\#, B \) and \( B^\# \) as bounded linear operators acting on \( \mathcal{R}(A) \oplus \mathcal{N}(A) \) have the forms \( A = A_1 \oplus 0 \),

\[
(2.7) A^\# = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ B_4 & B_2 \end{pmatrix}, \quad B^\# = \begin{pmatrix} (A_1B_1)^\#A_1 & C_3 \\ 0 & C_2 \end{pmatrix},
\]
respectively. Since $B A^*$ is group invertible, $B_2$ is group invertible and, hence, by
Theorem 2.5(i, ii), $B_1$ is group invertible, $B_2^* B_4 B_1^* = 0$ and

$$B^* = \begin{pmatrix} B_1 & 0 \\ B_4 & B_2 \end{pmatrix} = \begin{pmatrix} B_1^* \\ 0 \end{pmatrix} = \begin{pmatrix} B_2^* B_4 (B_1^*)^2 + (B_2^*)^2 B_4 B_1^* - B_2^* B_4 B_1^* \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (A_1 B_1)^* A_1 \\ 0 \end{pmatrix}.$$

It follows that $B_1^* = (A_1 B_1)^* A_1$ and $Y =: B_2^* B_4 (B_1^*)^2 + (B_2^*)^2 B_4 B_1^* - B_2^* B_4 B_1^* = 0.$

We have

$$\left\{ \begin{array}{c}
B_2^* Y B_1^* = 0, \\
B_4 B_2^* B_4 B_1^* = 0, \\
B_2 B_2^* B_4 B_1^* = 0 \\ 
\end{array} \right., \quad \Rightarrow \quad \left\{ \begin{array}{c}
B_2^* Y B_1^* = 0, \\
B_4 B_2^* B_4 B_1^* = 0, \\
B_2 B_2^* B_4 B_1^* = 0 \\ 
\end{array} \right..$$

Hence, $B_4 = 0$ and $B = B_1 \oplus B_2$.

(ii)$\Leftrightarrow$(iv): This is similar to the proof of (i)$\Leftrightarrow$(iii).

(iii)$\Leftrightarrow$(iv): Note that $A_1 B_1 \sim B_1 A_1.$ By Theorem 2.6, $A_1 B_1$ is group invertible if
and only if $B_1 A_1$ is group invertible. By Lemma 2.3, $(A_1 B_1)^* A_1 = A_1 (B_1 A_1)^*.$

Let dim$(\mathcal{M})$ denote the dimension of subspace $\mathcal{M}$. If dim$(\mathcal{R}(A))$ is finite, we
have the following result.

**Corollary 2.12.** Let $A, B, AB \in B(H)$ be group invertible and dim$(\mathcal{R}(A))$ be
finite. Then the following are equivalent:

(i) $(AB)^* = B^* A^*$;

(ii) $(BA)^* = A^* B^*$;

(iii) $\mathcal{R}(AB) = \mathcal{R}(BA)$ and $\mathcal{N}(AB) = \mathcal{N}(BA)$;

(iv) $\mathcal{R}(A^* B^*) = \mathcal{R}(B^* A^*)$ and $\mathcal{N}(A^* B^*) = \mathcal{N}(B^* A^*)$;

(v) $A = A_{11} \oplus A_{22} \oplus 0 \oplus 0, B = B_{11} \oplus 0 \oplus B_{33} \oplus 0,$ where $A_{11}, A_{22}, B_{11}$ and $B_{33}$
are invertible.

**Proof.** Obviously, (v)$\Rightarrow$(i)-(iv).

(i)$\Rightarrow$(v): From Theorem 2.11 we know that $A, A^*, B$ and $B^*$ as bounded linear
operators acting on $\mathcal{R}(A) \oplus \mathcal{N}(A)$ have the forms as in (2.7). If dim$(\mathcal{R}(A))$ is finite,
then (iii) implies that $B_1$ and $B_2$ are group invertible. From the proof of
Theorem 2.11((i)$\Rightarrow$(iii)), we get $B_2 = 0$. Hence, $A = A_1 \oplus 0, B = B_1 \oplus B_2$ and
$(A_1 B_1)^* = B_1^* A_1^{-1}$. Since $A_1 B_1 \sim B_1 A_1$ and $A_1 B_1$ is group invertible, by Theorem 2.11
$B_1 A_1$ is group invertible and $A_1^{-1} B_1^* = A_1^{-1} (A_1 B_1)^* A_1 = (B_1 A_1)^*$. 


A commuting group invertible operators is also group invertible. 

If operators acting on \( R \) is also an EP operator. The next result shows that the product of two self-adjoint operators acting on \( R \) is invertible. Hence, (v) holds.

Theorem 2.13. Let \( A, B \in B(H) \) be commutative closed range operators.

(i) If one of \( A \) and \( B \) is group invertible, then \( R(AB) \) is closed.

(ii) If \( A, B \) are group invertible, then \( AB \) is group invertible, and 

\[
(AB)^\# = B^\# A^\# = A^\# B^\# = (BA)^\#. 
\]

Proof. (i) Let \( A \) be group invertible and \( A \) have the form \( A = A_0 \oplus 0 \) with respect to the space decomposition \( H = R(A) \oplus N(A) \), where \( A_0 \) is invertible. Since \( AB = BA \), \( B \) can be written as \( B = B_0 \oplus B_{33} \), where \( B_0 \in B(R(A)) \) and \( B_{33} \in B(N(A)) \) and \( A_0 B_0 = B_0 A_0 \). Hence, \( R(B_0) = R(A) \cap R(B) \) is closed. Note that \( A_0 \) is invertible and \( R(A_0) = R(A) \). We have \( R(AB) = R(A_0 B_0) = R(B_0 A_0) = R(B_0) \).

(ii) In addition, if \( B = B_0 \oplus B_{33} \) is also group invertible, then \( B_0 \) and \( B_{33} \) are group invertible.
invertible. Similarly, $A_0B_0 = B_0A_0$ implies that that $A_0 = A_{11} \oplus A_{22}$, $B_0 = B_{11} \oplus 0$, with respect to the space decomposition $\mathcal{R}(A) = \mathcal{R}(B_0) \oplus \mathcal{N}(B_0)$, where $A_{ii} (i = 1, 2)$, $B_{11}$ are invertible with $A_{11}B_{11} = B_{11}A_{11}$. Now we have

$$
A = A_{11} \oplus A_{22} \oplus 0, \quad A^# = A_{11}^{-1} \oplus A_{22}^{-1} \oplus 0, \\
B = B_{11} \oplus 0 \oplus B_{33}, \quad B^# = B_{11}^{-1} \oplus 0 \oplus B_{33}^#.
$$

Thus, $(AB)^# = (A_{11}B_{11})^{-1} \oplus 0 \oplus 0 = B^#A^# = A^#B^# = (BA)^#$. \hfill \Box

We call $(A, B)$ is a commutative group invertible operator pair if $A, B \in \mathcal{B}(\mathcal{H})$ are group invertible and $AB = BA$. The following theorem has an independent interest.

**Theorem 2.14.** Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$, $(A, B)$ and $(C, D)$ be commutative group invertible operator pairs, respectively. For arbitrary $X \in \mathcal{B}(\mathcal{H})$,

$$AXC = BXD \iff A^#XC^# = B^#XD^#.$$  

**Proof.** Note that $(A^#)X(C^#)^# = AXC$ for every group invertible operator $A$ and $C$. To complete the proof, we only need to prove the necessity. Since $(A, B)$ is commutative group invertible operator pair, by (2.8),

$$A = A_{11} \oplus A_{22} \oplus 0, \quad A^# = A_{11}^{-1} \oplus A_{22}^{-1} \oplus 0, \\
B = B_{11} \oplus 0 \oplus B_{33}, \quad B^# = B_{11}^{-1} \oplus 0 \oplus B_{33}^#,
$$

where $A_{11}B_{11} = B_{11}A_{11}$. Similarly, since $(C, D)$ again is commutative group invertible operator pair,

$$C = C_{11} \oplus C_{22} \oplus 0, \quad C^# = C_{11}^{-1} \oplus C_{22}^{-1} \oplus 0, \\
D = D_{11} \oplus 0 \oplus D_{33}, \quad D^# = D_{11}^{-1} \oplus 0 \oplus D_{33}^#,
$$

where $C_{11}D_{11} = D_{11}C_{11}$. Let $X = (X_{ij})_{1 \leq i,j \leq 3}$. From $AXC = BXD$ we get

$$
\begin{pmatrix}
A_{11} & 0 & 0 \\
0 & A_{22} & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{pmatrix}
\begin{pmatrix}
C_{11} & 0 & 0 \\
0 & C_{22} & 0 \\
0 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
B_{11} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & B_{33}
\end{pmatrix}
\begin{pmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{pmatrix}
\begin{pmatrix}
D_{11} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & D_{33}
\end{pmatrix}.
$$

Then, by (2.11),

$$
\begin{pmatrix}
A_{11}X_{11}C_{11} & A_{11}X_{12}C_{22} & 0 \\
A_{22}X_{21}C_{11} & A_{22}X_{22}C_{22} & 0 \\
0 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
B_{11}X_{11}D_{11} & 0 & B_{11}X_{13}D_{33} \\
0 & 0 & 0 \\
B_{33}X_{31}D_{11} & 0 & B_{33}X_{33}D_{33}
\end{pmatrix}.
$$
Comparing the two sides of the above equation and, using the invertibility and commutativity of related operators, we have $X_{12} = 0$, $X_{21} = 0$, $X_{22} = 0$ and

\[
\begin{cases}
A_{11}X_{11}C_{11} = B_{11}X_{11}D_{11}, \\
X_{13}D_{33} = 0, \\
B_{33}X_{31} = 0, \\
B_{33}X_{33}D_{33} = 0.
\end{cases}
\] (2.12)

By (2.9)-(2.12), we have $A^#XC^# = B^#XD^#$. □

**Corollary 2.15.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be group invertible. For arbitrary $X \in \mathcal{B}(\mathcal{H})$, the following hold:

(i) $AX = XB \iff A^#X = XB^#$. (ii) $AXB = X \iff A^#XB^# = X$.

**3. The group invertibility of the anti-triangular operator matrices.** Let $A, B, D \in \mathcal{B}(\mathcal{H})$. The $2 \times 2$ operator matrices $M$ and $M_A$ are defined as

\[
M = \begin{pmatrix}
A & B \\
0 & D
\end{pmatrix} \quad \text{and} \quad M_A = \begin{pmatrix}
B & A \\
D & 0
\end{pmatrix}.
\] (3.1)

Then $M_A = M \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. It is clear that $M_A$ is obtained from $M$ by interchanging its columns. In order to consider the group invertibility of $M_A$, we can transfer our attention from $M_A$ to $M$. We need the next lemma. Recall that $\sigma(PQ) \setminus \{0\} = \sigma(QP) \setminus \{0\}$ (see [13, Remark 1.2.1]). It follows that $I + PQ$ is invertible if and only if $I + QP$ is invertible.

**Lemma 3.1.** (See [15, Corollary 1] and [14] for the matrix results) Let $A, P, Q \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{R}(A)$ is closed, $U = AQPAA^- + I - AA^-$. \hspace{1cm}

(i) If there exist $P_0, Q_0 \in \mathcal{B}(\mathcal{H})$ such that $P_0PA = A$ and $AQQ_0 = A$ and $U$ is invertible, then $T = PAQ$ is group invertible.

(ii) If $P$ and $Q$ are invertible, then $T = PAQ$ is group invertible $\iff U = AQPAA^- + I - AA^-$ is invertible.

**Proof.** (i) If $U = AQPAA^- + I - AA^- = I + (AQP - A)A^-$ is invertible, then $V =: I + A^- (AQP - A) = A^- AQP + I - A^- A$ is invertible. We get $UA = AV = AQP$ and $A = U^{-1} AQP = AQP V^{-1}$. Let $H = PU^{-1} P_0$, $G = Q_0 V^{-1} Q$ and $X = HTG$, 

where $P_0$ and $Q_0$ satisfy $P_0PA = A$ and $AQQ_0 = A$. From
\[ T = PAQ = P \begin{bmatrix} U^{-1}AQPA \end{bmatrix} Q = PU^{-1}P_0 \begin{bmatrix} PAQ \end{bmatrix} PAQ = HT^2 \]
and
\[ T = PAQ = P \begin{bmatrix} AQPAV^{-1} \end{bmatrix} Q = \begin{bmatrix} PAQ \end{bmatrix} \begin{bmatrix} PAQ \end{bmatrix} Q_0V^{-1}Q = T^2G, \]
we deduce that $TG = HT^2G = HT$ and $T^2G^2 = TG = HT = H^2T^2$. Hence,
\[ TX = THTG = T^2G^2 = H^2T^2 = HTGT = XT, \]
\[ TXT = T^2X = TT^2HTG = TTT^2G^2 = T^2G = T, \]
\[ XTX = H^2T^2X = HTX = HTHTG = HT^2G^2 = HTG = X, \]
i.e., $X$ is the group inverse of $T$. Moreover,
\[ T^# = HTG = PU^{-1}P_0PAQQ_0V^{-1}Q = PU^{-1}AV^{-1}Q = PU^{-2}AQ = PAV^{-2}Q. \]

(ii) By (i) we know that $T = PAQ$ is group invertible if $U$ is invertible. Now we prove that $U$ is invertible if $T = PAQ$ is group invertible. Note that $A$ is an operator from $\mathcal{R}(A^*) \oplus \mathcal{N}(A)$ into $\mathcal{R}(A) \oplus \mathcal{N}(A^*)$. $P$ is an operator from $\mathcal{R}(A) \oplus \mathcal{N}(A^*)$ into $\mathcal{H}$, $Q$ is an operator from $\mathcal{H}$ into $\mathcal{R}(A^*) \oplus \mathcal{N}(A)$ having the forms
\[ A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} P_1 & P_2 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & Q_2 \\ 0 & 0 \end{pmatrix}, \]
respectively, where $A_1 \in \mathcal{B}(\mathcal{R}(A^*), \mathcal{R}(A))$ is invertible. Since $P, Q$ are invertible and $A^-$ exists, we have the representations
\[ P^{-1} = \begin{pmatrix} P_1' & P_2' \\ P_1' & P_2' \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} Q_1' & Q_2' \\ Q_1' & Q_2' \end{pmatrix}, \quad A^- = \begin{pmatrix} A_1^{-1} & A_3' \\ A_4 & A_2' \end{pmatrix} \]
and
\[ (3.2) \quad \begin{pmatrix} P_1' & P_1' \\ P_2' & P_2' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} Q_1' & Q_1' \\ Q_2' & Q_2' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \]
where $A_i', i = 2, 3, 4$ are arbitrary. Then $T = PAQ = P_1A_1Q_1$,
\[ (3.3) \quad U = AQPAA^- + I - AA^- = \begin{pmatrix} A_1Q_1P_1 & [A_1Q_1P_1 - I]A_1A_3' \\ P_1A_1Q_1 & 0 \end{pmatrix}. \]
Let $X$ be the group inverse of $T$. Then
\[ (a) P_1A_1Q_1XP_1A_1Q_1 = P_1A_1Q_1, \quad (b) P_1A_1Q_1X = XP_1A_1Q_1, \quad (c) XP_1A_1Q_1X = X. \]
Pre-multiplying (a) by $A^{-1}_1P'_1$ and then post-multiplying by $Q'_1A^{-1}_1$, and then applying (3.2) gives $Q_1XP_1 = A^{-1}_1$. Analogously, by (3.2) and (b) we get $Q_1X = A^{-1}_1P'_1XP_1A_1Q_1$. Hence, $P'_1XP_1A_1Q_1P_1 = I$, which implies that $A_1Q_1P_1$ is left invertible. Note that (a) and (b) imply that $P'_1A_1Q_1XP_1A_1Q_1 = P_1A_1Q_1XP_1A_1Q_1 = P_1A_1Q_1$, By (3.2) we get $A_1Q_1P_1A_1Q_1XP_1A_1Q_1 = I$, which implies that $A_1Q_1P_1$ is right invertible. Hence, $A_1Q_1P_1$ is invertible. Note that (a) and (b) imply that $P'_1A_1Q_1XP_1A_1Q_1P_1 = I$, which implies that $A_1Q_1P_1$ is invertible. Hence, $A_1Q_1P_1$ is invertible and by (3.3) we have $U$ is invertible.

For EP operators we have the following results.

**Theorem 3.2.** Let $M$ and $M_A$ be defined as in (3.1) and $A, D$ be EP operators. Then (i)

(a) $M$ is an EP operator

(b) $A^*BD^* = 0$ and $A^*BD^* + A^*BD^* = 0$

(c) $B = AA^*B$ and $B = BDD^*$

(d) $R(B) \subset R(A)$ and $N(D) \subset N(B)$.

In this case,

$$M^+ = M^# = \begin{pmatrix} A^# & -A^*BD^* \\ 0 & D^* \end{pmatrix}.$$

(ii) $M$ is an EP operator and $A^* = D^*$ if and only if $M_A$ is an EP operator and $M_A^# = \begin{pmatrix} 0 & D^* \\ A^# & -A^*BD^* \end{pmatrix}$.

**Proof.** (i) (a) ⇒ (b): By Theorem 2.5 (ii), if $M$ is an EP operator, then $A^*BD^* = 0$ and

$$MM^# = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A^# & (A^#)^2BD^* + A^*B(D^#)^2 - A^*BD^* \\ 0 & D^* \end{pmatrix}$$

$$= \begin{pmatrix} AA^# & A^*BD^* + A^*BD^* \\ 0 & DD^* \end{pmatrix}$$

is selfadjoint. It follows

$$A^*BD^* + A^*BD^* = 0. \tag{3.4}$$

(b) ⇒ (c): Post-multiplying (3.4) by $D$ gives $A^*B = 0$. Similarly, $BD^* = 0$.

(c) ⇒ (d): See Lemma 2.1.

(c) ⇒ (a): By Theorem 2.5 (ii), $M^# = \begin{pmatrix} A^# & -A^*BD^* \\ 0 & D^* \end{pmatrix}$. Since $A, D$ are EP operators, it is straightforward to check that $M^#$ satisfies (I, II, III, IV, V).
(ii) Necessity: \( A \) and \( D \) are EP operators and \( X = \begin{pmatrix} 0 & D^* \\ A^* & -A^* BD^* \end{pmatrix} \). By (i)(c), it is easy to check \( XM_A = DD^* \oplus AA^*, M_A X = AA^* \oplus DD^*, M_X M_A = M_A \) and \( X M_A X = X \). Now, \( A, D \) are EP operators and \( A^\pi = D^\pi \) imply that \( M_A \) is an EP operator.

Sufficiency: If \( M_A^# = \begin{pmatrix} 0 & D^* \\ A^* & -A^* BD^* \end{pmatrix} \), then
\[
M_A M_A^# = \begin{pmatrix} AA^* & A^* BD^* \\ 0 & DD^* \end{pmatrix} \quad \text{and} \quad M_A^# M_A = \begin{pmatrix} DD^* & 0 \\ A^* BD^* & AA^* \end{pmatrix}.
\]
Since \( M_A \) is an EP operator, \( M_A M_A^# = M_A^# M_A \). So \( A^\pi = D^\pi, A^\pi BD^* = 0 \) and \( A^\pi BD^\pi = 0 \). Since
\[
MM_A^#M = \begin{pmatrix} B & A \\ D & 0 \end{pmatrix} \begin{pmatrix} DD^* & 0 \\ A^* BD^* & AA^* \end{pmatrix} = \begin{pmatrix} BDD^* - AA^* BD^\pi & A \\ D & 0 \end{pmatrix} \quad \text{and} \quad M_A^2 = \begin{pmatrix} BD^* D & A \\ D & 0 \end{pmatrix} = M,
\]
we have \( BD^* D = B \). Hence, \( A^\pi B = A^\pi BD^* D = 0 \). By (i)(c) of this theorem we arrive the result.

Theorem 3.2 (ii), the group invertibility of \( M \) doesn’t imply that \( M_A \) is group invertible in general. We can see from the following example.

**Example 3.3.** Define \( M \) on \( \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \) by
\[
M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad D = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.
\]
Then \( M \) is group invertible by Theorem 2.5(ii) (in fact \( A^\pi BD^\pi = 0 \)). Since
\[
M_A = \begin{pmatrix} B & A \\ D & 0 \end{pmatrix} = \begin{pmatrix} 0 & I & I \\ -I & 0 & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -I & -I & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]
\( \mathcal{R}(M_A) \neq \mathcal{R}(M_A^2) \), hence \( M_A \) is not group invertible.

Let \( T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \). Then \( \begin{pmatrix} C & A \\ B & 0 \end{pmatrix} = T \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \). Theorem 3.1(ii) shows that, if \( \mathcal{R}(T) \) is closed, then anti-triangular operator matrix \( \begin{pmatrix} C & A \\ B & 0 \end{pmatrix} \)
is group invertible if and only if $U = T \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} TT^+ + I - TT^+$ is invertible. We have the following interesting cases.

**THEOREM 3.4.** Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed. Let $c_1, c_2 \in \mathbb{C}$ and $k, l$ be positive integers.

(i) If $A$ is invertible, then $\begin{pmatrix} C & A \\ B & 0 \end{pmatrix}$ is group invertible if and only if $C(I - B^+B) - AB$ is invertible.

(ii) $\begin{pmatrix} c_1A + c_2B & A \\ B & 0 \end{pmatrix}$ is group invertible if and only if

\[
\begin{pmatrix} c_1A^2A^+ + I - AA^+ \\
BAA^+ \\
(1 + c_1c_2)ABB^+ \\
c_2B^2B^+ + I - BB^+ \end{pmatrix}
\]

is invertible.

(iii) $\begin{pmatrix} A^kB^l & A \\ B & 0 \end{pmatrix}$ is group invertible if and only if

\[
\begin{pmatrix} I - AA^+ & ABB^+ \\
BAA^+ & BA^kB^lB^+ + I - BB^+ \end{pmatrix}
\]

is invertible.

(iv) If $A^2 = A$, then $\begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$ is group invertible if and only if $I - BB^+ - BAB^+$ is invertible.

**Proof.** (i) Let $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. If $A$ is invertible, then it is easy to get $T^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^+ \\ 0 & B^+ \end{pmatrix}$ and $TT^+ = \begin{pmatrix} I & 0 \\ 0 & BB^+ \end{pmatrix}$. By Theorem 3.1(ii), $\begin{pmatrix} C & A \\ B & 0 \end{pmatrix}$ is group invertible if and only if

\[
U = TT^+ + I - TT^+ = \begin{pmatrix} C & A \\ B & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & BB^+ \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I - BB^+ \end{pmatrix}
\]

(3.5)
is invertible. Note that
\[
\begin{pmatrix}
I & -(CB+ABB^+) \\
0 & I
\end{pmatrix}
\begin{pmatrix}
C & ABB^+ \\
B & I-BB^+
\end{pmatrix}
\begin{pmatrix}
I & B^+ \\
0 & I
\end{pmatrix}
= 
\begin{pmatrix}
C-CB^+B-AB & 0 \\
B & I
\end{pmatrix}.
\]
We get U is invertible if and only if \(C(I-B^+B)-AB\) is invertible.

(ii) Since \(\mathcal{R}(A)\) and \(\mathcal{R}(B)\) are closed, \(A^+\) and \(B^+\) exist and it is easy to check that \(\begin{pmatrix} 0 & A^+ \\ A^+ & B^+ \end{pmatrix}\) is a \(\{1\}\)-inverse of \(\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\). Note that
\[
\begin{pmatrix}
c_1A + c_2B & A \\
B & 0
\end{pmatrix}
= 
\begin{pmatrix}
I & c_2I \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
c_1I & I
\end{pmatrix}.
\]
By Theorem 3.1(ii) again, \(\begin{pmatrix} c_1A + c_2B & A \\ B & 0 \end{pmatrix}\) is group invertible if and only if
\[
U = 
\begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
c_1I & I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
B & 0
\end{pmatrix}
\begin{pmatrix}
0 & A^+ \\
A^+ & B^+
\end{pmatrix}
+ I - \begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
\begin{pmatrix}
0 & B^+ \\
A^+ & 0
\end{pmatrix}
= 
\begin{pmatrix}
c_1A + (1+c_1c_2)A \\
B c_2B
\end{pmatrix}
\begin{pmatrix}
AA^+ & 0 \\
0 & BB^+
\end{pmatrix}
+ \begin{pmatrix}
I & 0 \\
0 & I-BB^+
\end{pmatrix}
= 
\begin{pmatrix}
c_1A^2A^+ + I - AA^+ \\
BAA^+
\end{pmatrix}
\begin{pmatrix}
(1+c_1c_2)ABB^+
\end{pmatrix}
= 
\begin{pmatrix}
c_1A^2 + A^+ \\
BAA^+
\end{pmatrix}
\begin{pmatrix}
1+c_1c_2ABB^+
\end{pmatrix}.
\]
(iii) Note that
\[
\begin{pmatrix}
A^kB^l & A \\
B & 0
\end{pmatrix}
= 
\begin{pmatrix}
I & A^kB^{l-1} \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}.
\]
We know \(\begin{pmatrix} A^kB^l & A \\ B & 0 \end{pmatrix}\) is group invertible if and only if
\[
U = 
\begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
\begin{pmatrix}
I & A^kB^{l-1} \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
\begin{pmatrix}
0 & B^+ \\
A^+ & 0
\end{pmatrix}
+ I - \begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
\begin{pmatrix}
0 & B^+ \\
A^+ & 0
\end{pmatrix}
= 
\begin{pmatrix}
0 & A \\
B & BA^kB^{l-1}
\end{pmatrix}
\begin{pmatrix}
AA^+ & 0 \\
0 & BB^+
\end{pmatrix}
+ \begin{pmatrix}
I & 0 \\
0 & I-BB^+
\end{pmatrix}
= 
\begin{pmatrix}
I & A^+ \\
BAA^+ & BA^kB^{l-1} + I - BB^+
\end{pmatrix}.
(iv) Let $c_1 = 1$ and $c_2 = 0$ in (ii). Then \( \begin{pmatrix} A & A \\ B & 0 \end{pmatrix} \) is group invertible if and only if

\[
\begin{pmatrix}
I & ABB^+ \\
BAA^+ & I - BB^+
\end{pmatrix}
\]

is invertible, which is equivalent to the Schur complement $I - BB^+ - BABB^+$ is invertible.

REFERENCES