Electronic Journal of Linear Algebra

Volume 32 *Volume 32 (2017)* Article 32

2017

On the distance signless Laplacian spectral radius of graphs and digraphs

Dan Li
Xinjiang University,Urumqi, ldxjedu@163.com

Guoping Wang
Xinjiang Normal University,Urumqi, xj.wgp@163.com

Jixiang Meng
Xinjiang University, Urumqi, mjxxju@sina.com

Recommended Citation

Li, Dan; Wang, Guoping; and Meng, Jixiang. (2017), "On the distance signless Laplacian spectral radius of graphs and digraphs", *Electronic Journal of Linear Algebra*, Volume 32, pp. 438-446.
DOI: https://doi.org/10.13001/1081-3810.1982

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 438-446, December 2017.
http://repository.uwyo.edu/ela

ON THE DISTANCE SIGNLESS LAPLACIAN SPECTRAL RADIUS OF GRAPHS AND DIGRAPHS*

DAN LI†, GUOPING WANG‡, AND JIXIANG MENG†

Abstract. Let $\eta(G)$ denote the distance signless Laplacian spectral radius of a connected graph G. In this paper, bounds for the distance signless Laplacian spectral radius of connected graphs are given, and the extremal graph with the minimal distance signless Laplacian spectral radius among the graphs with given vertex connectivity and minimum degree is determined. Furthermore, the digraph that minimizes the distance signless Laplacian spectral radius with given vertex connectivity is characterized.

Key words. Distance signless Laplacian spectral radius, Bound, Extremal graph, Digraph, Vertex connectivity.

AMS subject classifications. 05C50.

1. Introduction. In this paper, we consider simple and connected graphs. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let d_i be the degree of the vertex v_i in G for $i = 1, \ldots, n$. Let $A(G) = (a_{ij})_{n\times n}$ be the $(0,1)$-adjacency matrix of G, where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise. Let $\triangle(G) = \mathrm{diag}(d_1, \ldots, d_n)$ be the diagonal degree matrix. Then $Q(G) = \triangle(G) + A(G)$ is the signless Laplacian matrix of G.

Let $D(G) = (d_{ij})_{n\times n}$ be the distance matrix of a connected graph G, where $d_{ij} = d_G(v_i, v_j)$ is defined to be the length of shortest path between v_i and v_j. We call $D_i = \sum_{j=1}^{n} d_{ij}$ the *transmission* of vertex v_i $(i = 1, 2, \ldots, n)$. The matrix $D^Q(G) = \mathrm{Diag}(Tr) + D(G)$ is the *distance signless Laplacian matrix* of G (D^Q-matrix), where $\mathrm{Diag}(Tr)$ is the diagonal matrix whose ith diagonal entry is D_i. The matrix $D^Q(G)$ is nonnegative and irreducible when G is connected. The largest eigenvalue of $D^Q(G)$ is called *distance signless Laplacian spectral radius* of the graph G.

The distance spectral radius of a connected graph has been studied extensively. Chen, Lin and Shu [4] obtained sharp upper bounds on the distance spectral radius of a graph. Indulal [7] gave some bounds for the distance spectral radius of graphs. Ilić [6] obtained the tree with given matching number that minimizes the distance spectral radius. Bose, Nath and Paul [3] determined the unique graph with maximal distance spectral radius in the class of graphs without a pendent vertex. Lin, Yang, Zhang and Shu [10] characterized the extremal digraph with the minimal distance spectral radius among all digraphs with given vertex connectivity and the extremal graph with the minimal distance spectral radius among all graphs with given edge connectivity.

Aouchiche and Hansen [1] introduced the distance Laplacian and distance signless Laplacian spectra of graphs, respectively. Lin and Lu [9] found a sharp lower bound as well as a sharp upper bound of the distance signless Laplacian spectral radius in terms of the clique number. Furthermore, both extremal graphs are

*Received by the editors on June 7, 2017. Accepted for publication on November 13, 2017. Handling Editor: Ravi Bapat.
†College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, China (ldxjedu@163.com, mjxxju@sina.com).
‡School of Mathematical Sciences, Xinjiang Normal University, Urumqi, Xinjiang 830054, China (xj.wgp@163.com).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 438-446, December 2017.
http://repository.uwyo.edu/ela

uniquely determined. Hong and You [5] obtained some sharp bounds on the distance signless Laplaciqan spectral radius of graphs. Xing and Zhou [11] gave the unique graphs with minimum distance and distance signless Laplacian spectral radius among bicyclic graphs with fixed number of vertices. Xing, Zhou and Li [12] determined the graphs with minimum distance signless Laplacian spectral radius among the trees, unicyclic graphs, bipartite graphs, the connected graphs with fixed pendant vertices and fixed connectivity, respectively.

In this paper, we obtain the bounds for the distance signless Laplacian spectral radius of graphs and determine the extremal graph with the minimum distance signless Laplacian spectral radius among the graphs with given vertex connectivity and minimum degree. Furthermore, we characterize the digraph that minimizes the distance signless Laplacian spectral radius with given vertex connectivity.

2. Bounds on the distance signless Laplacian spectral radius of graphs. Let G be a graph. We denote the distance signless Laplacian spectral radius of G by $\eta(G)$. A graph G is *transmission regular* if $D_1 = D_2 = \cdots = D_n$.

Lemma 2.1. [5] *Let G be a connected graph on n vertices. Then*

$$\min\{D_i : 1 \le i \le n\} \le \frac{\eta(G)}{2} \le \max\{D_i : 1 \le i \le n\}.$$

Moreover, one of the equalities holds if and only if G is a transmission regular graph.

Let G be a graph on n vertices. It was shown in [11] that $\eta(G) \ge \frac{2}{n}\sum_{i=1}^{n} D_i$ with equality if and only if G is transmission regular.

Note that $\sum_{i=1}^{n} D_i^2 \ge \frac{1}{n}(\sum_{i=1}^{n} D_i)^2$. Then $2\sqrt{\frac{D_1^2+D_2^2+\cdots+D_n^2}{n}} \ge \frac{2}{n}\sum_{i=1}^{n} D_i$. Thus, the following result gives a better lower bound of $\eta(G)$.

Theorem 2.2. *Let G be a graph on n vertices. Then $\eta(G) \ge 2\sqrt{\frac{D_1^2+D_2^2+\cdots+D_n^2}{n}}$ with equality if and only if G is transmission regular.*

Proof. Let $\mathbf{x} = \frac{1}{\sqrt{n}}(1,1,\ldots,1)$. Then $\eta(G) \ge \sqrt{\mathbf{x}(D^Q(G))^2\mathbf{x}^T}$.

Note that

$$\mathbf{x}D^Q(G) = \frac{1}{\sqrt{n}}(1,1,\ldots,1)D^Q(G) = \frac{2}{\sqrt{n}}(D_1, D_2, \ldots, D_n).$$

We have

$$\mathbf{x}(D^Q(G))^2\mathbf{x}^T = \mathbf{x}D^Q(G)(\mathbf{x}D^Q(G))^T = \frac{4}{n}\sum_{i=1}^{n} D_i^2.$$

Thus, $\eta(G) \ge 2\sqrt{\frac{D_1^2+D_2^2+\cdots+D_n^2}{n}}$.

If G is transmission regular, then by Lemma 2.1, $\eta(G) = 2D_i = 2\sqrt{\frac{1}{n}\sum_{i=1}^{n} D_i^2}$.

Conversely, if equality holds, then $\mathbf{x}$ is the eigenvector corresponding to $\eta(G)$, i.e., $D^Q(G)\mathbf{x}^T = \eta(G)\mathbf{x}^T$. This implies that $D_i = \eta(G)/2$ $(i = 1, 2, \ldots, n)$. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 438-446, December 2017.
http://repository.uwyo.edu/ela

Let $\rho(G)$ be the distance spectral radius of the graph G. If G is transmission regular, then $\rho(G) = D_i$. Thus, we have the following.

COROLLARY 2.3. *Suppose that G is transmission regular. Then $\eta(G) = 2\rho(G)$.*

A graph G is *regular* if $d_1 = \cdots = d_n$.

LEMMA 2.4. *Let G be a connected graph with diameter $d \leq 2$. Then G is regular if and only if G is transmission regular.*

Proof. If $d = 1$, then $G \cong K_n$. If $d = 2$, then $D_i = d_i + 2(n-1-d_i) = 2n-2-d_i$, from which we can see that G is regular if and only if G is transmission regular. □

The unit eigenvector $\mathbf{x}(G) = (x_1, x_2, \dots, x_n)^T > 0$ corresponding to $\eta(G)$ is called the *Perron vector* of $D^Q(G)$.

THEOREM 2.5. *Let G be a graph on n vertices with maximum degree Δ_1 and second maximum degree Δ_2. Then $\eta(G) \geq 4n-4-\Delta_1-\Delta_2$ with equality if and only if G is a regular graph with diameter $d \leq 2$.*

Proof. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ be a Perron eigenvector of $D^Q(G)$ such that $x_i = \min_{1\leq k\leq n} x_k$ and $x_j = \min_{1\leq k\neq i\leq n} x_k$.

From $D^Q(G)\mathbf{x} = \eta(G)\mathbf{x}$, we obtain that

$$\eta(G)x_i = D_i x_i + \sum_{k=1,k\neq i}^{n} d_{ik}x_k \geq D_i(x_i + x_j);$$

$$\eta(G)x_j = D_j x_j + \sum_{k=1,k\neq j}^{n} d_{jk}x_k \geq D_j(x_i + x_j).$$

Note that

$$D_i = \sum_{k=1,k\neq i}^{n} d_{ik} \geq d_i + (n-1-d_i)2 = 2n-2-d_i;$$

$$D_j = \sum_{k=1,k\neq j}^{n} d_{jk} \geq d_j + (n-1-d_j)2 = 2n-2-d_j.$$

Then we can get that

$$\eta(G)x_i \geq (2n-2-d_i)(x_i + x_j) \tag{2.1}$$

and

$$\eta(G)x_j \geq (2n-2-d_j)(x_i + x_j). \tag{2.2}$$

Therefore, by (2.1) and (2.2), we have

$$\eta(G)(x_i + x_j) \geq (4n-4-d_i-d_j)(x_i + x_j) \geq (4n-4-\Delta_1-\Delta_2)(x_i + x_j),$$

that is

$$\eta(G) \geq 4n-4-\Delta_1-\Delta_2.$$

If the equality holds, then we deduce that G is a regular graph with diameter $d \leq 2$ and all x_i are equal. If $d = 1$, then $G \cong K_n$. If $d = 2$, then we get $\eta(G)x_i = 2(d_i + (n-1-d_i)2)x_i$, i.e., $\eta(G) = 2(2n-2-d_i)$, which implies that G is a regular graph. Conversely, by Lemma 2.4, we can get that $\eta(G) = 2(2n-2-\Delta_1)$ if G is a regular graph with diameter $d \leq 2$. □

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 438-446, December 2017.
http://repository.uwyo.edu/ela

THEOREM 2.6. *Let G be a graph on n vertices with minimum degree δ_1 and second minimum degree δ_2. Then $\eta(G) \le 2dn - d(d-1) - 2 - (d-1)(\delta_1+\delta_2)$ with equality if and only if G is a regular graph with diameter $d \le 2$.*

Proof. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ be a Perron eigenvector of $D^Q(G)$ such that $x_s = \max_{1\le k\le n} x_k$ and $x_t = \max_{1\le k\ne s\le n} x_k$.

From $D^Q(G)\mathbf{x} = \eta(G)\mathbf{x}$, we obtain that

$$\eta(G)x_s = D_s x_s + \sum_{k=1,k\ne s}^{n} d_{sk}x_k \le D_s(x_s+x_t);$$

$$\eta(G)x_t = D_t x_t + \sum_{k=1,k\ne t}^{n} d_{tk}x_k \le D_t(x_s+x_t).$$

Note that

$$D_s = \sum_{k=1,k\ne s}^{n} d_{sk} \le d_s + 2 + \cdots + d - 1 + d[n-1-d_s-(d-2)] = dn - \frac{d(d-1)}{2} - 1 - d_s(d-1).$$

Thus,

$$\eta(G)x_s \le [dn - \frac{d(d-1)}{2} - 1 - d_s(d-1)](x_s+x_t). \tag{2.3}$$

Similarly, we can get that

$$\eta(G)x_t \le [dn - \frac{d(d-1)}{2} - 1 - d_t(d-1)](x_s+x_t). \tag{2.4}$$

Therefore, by (2.3) and (2.4), we have

$$\eta(G)(x_s+x_t) \le [2dn - d(d-1) - 2 - (d-1)(d_s+d_t)](x_s+x_t),$$

from which we obtain that $\eta(G) \le 2dn - d(d-1) - 2 - (d-1)(\delta_1+\delta_2)$.

If the equality holds, then all x_i are equal, and hence, $D^Q(G)$ has equal row sums. Therefore, G is transmission regular. If $d \ge 3$, then by (2.3), we know that for each vertex v_i, there is exactly one vertex v_j such that $d_{ij} = 2$, and then $d < 4$. If the diameter of G is 3 and equality holds, then for a center vertex v_s, from $D^Q(G)\mathbf{x} = \eta(G)\mathbf{x}$ and (2.3), written for the component x_s, we have

$$\eta(G)x_s = D_s x_s + d_s x_s + (n-1-d_s)2x_s = 2(2n-2-d_s)x_s = 2(3n-4-2d_s)x_s,$$

and thus, $d_s = n-2$, which implies that $G \cong P_4$, but P_4 is not transmission regular, a contradiction. Therefore, G is a regular graph with diameter $d \le 2$. Conversely, by Lemma 2.4, we can get that $\eta(G) = 2(2n-2-\delta_1)$ if G is a regular graph with diameter $d \le 2$. ∎

3. The minimal distance signless Laplacian spectral radius for graphs with given vertex connectivity and minimum degree. The *vertex connectivity* of a graph G is the minimum number of vertices, whose deletion results in a disconnected or a trivial graph. The *join* $G_1 + G_2$ of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ consists of the union $G_1 \cup G_2$ of G_1 and G_2 and all edges joining V_1 with V_2. Let

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 438-446, December 2017.
http://repository.uwyo.edu/ela

$\mathcal{G}_{k,n}$ be the class of all graphs of order n with vertex connectivity $\kappa(G) \leq k \leq n-1$ and minimum degree $\delta \geq k$. Let $G_{k,\delta,n} = K_k + (K_{\delta-k+1} \cup K_{n-\delta-1})$. Obviously, $G_{k,\delta,n} \in \mathcal{G}_{k,n}$.

For fixed n and k, let $g(n_1,n_2) = x^3 - a_1x^2 - a_2x + a_3$, where $a_1 = 4n + n_1 + n_2 - 6$, $a_2 = (n_1+n_2)(20-13k) - 8(n_2^2+n_1^2) - 12(n_1n_2+1) - 5k^2 + 16k$, and $a_3 = -2k^3 + (10-8n_2-8n_1)k^2 + (-16+26n_2+26n_1-10n_2^2-10n_1^2-16n_1n_2)k - 4((n_2-2)^2 + (n_1-2)^2 + (n_1+n_2)(2n_1n_2+1) + 2(3n_1n_2+1))$.

Lemma 3.1. *$\eta = \eta(K_k + (K_{n_1} \cup K_{n_2}))$ satisfies the equation*

$$g(n_1,n_2) = 0.$$

Proof. Let $\mathbf{x}$ be the Perron vector of $D^Q(K_k + (K_{n_1} \cup K_{n_2})$. It is clear that $\mathbf{x}$ can be written as

$$\mathbf{x} = (\underbrace{x,\ldots,x}_{n_1}, \underbrace{y,\ldots,y}_{k}, \underbrace{z,\ldots,z}_{n_2})^T.$$

From $D^Q(K_k + (K_{n_1} \cup K_{n_2})\mathbf{x} = \eta\mathbf{x}$, we have

$$\eta x = (2n_1 + 2n_2 + k - 2)x + ky + 2n_2z;$$

$$\eta y = n_1x + (n_1 + n_2 + 2k - 2)y + n_2z;$$

$$\eta z = 2n_1x + ky + (2n_1 + 2n_2 + k - 2)z.$$

This shows that η is an eigenvalue of the matrix A, where

$$A = \begin{pmatrix} 2n_1+2n_2+k-2 & k & 2n_2 \\ n_1 & n_1+n_2+2k-2 & n_2 \\ 2n_1 & k & 2n_1+2n_2+k-2 \end{pmatrix},$$

and $det(Ix - A) = g(n_1,n_2) = x^3 - a_1x^2 - a_2x + a_3$. This shows that the result is true. ∎

By Lemma 3.1 we can immediately get the following corollary.

Corollary 3.2. *Spectral radius of $G_{k,\delta,n}$ is the largest root of the equation*

$$\begin{aligned} &x^3 - (5n-k-6)x^2 - (-4\delta^2 + (4k+4n-8)\delta - nk - 8n^2 + 24n - 16)x + ((-12k+8n+16)\delta^2 \\ &+ (12k^2 + 4kn - 8n^2 - 40k + 32)\delta - 4k^3 - 2kn^2 - 8k^2 + 54kn - 12n^2 - 56k + 4n - 24) = 0. \end{aligned}$$

Lemma 3.3. *If $n_2 \geq n_1 \geq 2$, then $\eta(K_k + (K_{n_1} \cup K_{n_2})) > \eta(K_k + (K_{n_1-1} \cup K_{n_2+1}))$.*

Proof. Let $h(\lambda) = g_{n_1,n_2}(\lambda) - g_{n_1-1,n_2+1}(\lambda) = -4(n_2-n_1+1)\lambda + 4(k(n_2-n_1+1) + (n_2-n_1)(n_2+n_1-1) + 2(n_1-1))$. Then $\lambda_0 = k + \frac{(n_2-n_1)(n_2+n_1-1)}{n_2-n_1+1} + \frac{2(n_1-1)}{n_2-n_1+1}$ is the root of the equation $h(\lambda) = 0$.

By Lemma 2.1, we know that $\eta(K_k + (K_{n_1} \cup K_{n_2})) \geq 2(n_2+n_1-1) + 2k$ and $\eta(K_k + (K_{n_1-1} \cup K_{n_2+1})) \geq 2(n_2+n_1-1) + 2k$. And,

$$\begin{aligned} 2(n_2+n_1-1+k) - \lambda_0 &= 2(n_2+n_1-1) + k - \frac{(n_2-n_1)(n_2+n_1-1)}{n_2-n_1+1} - \frac{2(n_1-1)}{n_2-n_1+1} \\ &= \left(1 - \frac{n_2-n_1}{n_2-n_1+1}\right)(n_2+n_1-1) + \frac{(n-1)(n_2-n_1+1) - 2n_1 + 2}{n_2-n_1+1} > 0. \end{aligned}$$

Thus, $\eta(K_k + (K_{n_1} \cup K_{n_2})) > \lambda_0$ and $\eta(K_k + (K_{n_1-1} \cup K_{n_2+1})) > \lambda_0$. Since $h(\lambda)$ is a decreasing function of λ, $g_{n_1,n_2}(\lambda) - g_{n_1-1,n_2+1}(\lambda) < 0$ for $\lambda > \lambda_0$, which yields $\eta(K_k + (K_{n_1} \cup K_{n_2})) > \eta(K_k + (K_{n_1-1} \cup K_{n_2+1}))$. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 438-446, December 2017.
http://repository.uwyo.edu/ela

LEMMA 3.4. [11] *Let G be a connected graph with $u, v \in V(G)$ and $uv \notin E(G)$. Then $\eta(G) > \eta(G+uv)$.*

THEOREM 3.5. *Among all the connected graphs of order n with connectivity at most k and minimum degree δ, $G_{k,\delta,n}$ uniquely minimizes the distance signless Laplacian spectral radius.*

Proof. The case $k = n-1$ is trivial, and so we assume that $k \leq n-2$. Let G be a graph in $\mathcal{G}_{k,n}$ with the minimum distance signless Laplacian spectral radius, and S be a k-vertex cut of G. Now we verify the following two claims.

CLAIM 1. *$G - S$ consists exactly of two components.*

Suppose to the contrary that $G - S$ contains G_1, G_2 and G_3. Then there must be $u \in G_2$ and $v \in G_3$ such that $uv \notin E(G)$. Thus, by Lemma 3.4, $\eta(G) > \eta(G+uv)$ while $G + uv \in \mathcal{G}_{k,n}$. This contradiction shows that $G - S = G_1 \cup G_2$.

CLAIM 2. *G_i and $G[V(G_i) \cup S]$ $(i = 1, 2)$ are all complete graphs.*

If there is $uv \notin E(G_i)$, then $G+uv \in \mathcal{G}_{k,n}$. By Lemma 3.4, $\eta(G) > \eta(G+uv)$, a contradiction. Similarly, $G[V(G_i) \cup S]$ is also a complete graphs.

By the above argument we can determine that $G \cong K_k + (K_{n_1} \cup K_{n_2})$. Since $\delta \geq k$, we have $n_i \geq \delta - k + 1$. If $n_1 \geq \delta - k + 2$, without loss of generality, we suppose $n_2 \geq n_1$, then $\eta(G) > \eta(K_k + (K_{n_1-1} \cup K_{n_2+1}))$ while $K_k + (K_{n_1-1} \cup K_{n_2+1}) \in \mathcal{G}_{k,n}$ by Lemma 3.3. This contradiction shows that $n_1 = \delta - k + 1$ and $n_2 = n - \delta - 1$, that is, $G \cong G_{k,\delta,n}$. $\square$

Let $\mathcal{C}(n, k)$ be the set of connected graphs with n vertices and connectivity k, where $1 \leq k \leq n-2$, then the following result can be obtained immediately.

COROLLARY 3.6. [12] *Let $G \in \mathcal{C}(n, k)$,where $1 \leq k \leq n-2$. Then, $\eta(G) \geq \eta(K_k + (K_1 \cup K_{n-k-1}))$ with equality if and only if $G = K_k + (K_1 \cup K_{n-k-1})$.*

4. The minimal distance signless Laplacian spectral radius for digraphs with given vertex connectivity. Similar to undirected graph, we use $D(\overrightarrow{G}) = (d_{ij})_{n \times n}$ and $D^Q(\overrightarrow{G}) = \mathrm{Diag}(Tr) + D(\overrightarrow{G})$ to denote the *distance matrix* and *distance signless Laplacian matrix* of $\overrightarrow{G}$, respectively, where $d_{ij} = d_{\overrightarrow{G}}(v_i, v_j)$ denote the length of shortest dipath from v_i to v_j in $\overrightarrow{G}$ and $\mathrm{Diag}(Tr)$ is the diagonal matrix with D_i. Clearly, the matrix $D^Q(\overrightarrow{G})$ is irreducible when $\overrightarrow{G}$ is strongly connected. The eigenvalue of $D^Q(\overrightarrow{G})$ with the largest modulus is the *distance signless Laplacian spectral radius* of $\overrightarrow{G}$, denoted by $\eta(\overrightarrow{G})$. For a strongly connected digraph $\overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G}))$, a set of vertices $S \subset V(\overrightarrow{G})$ is a vertex cut if $\overrightarrow{G} - S$ is not strongly connected. The *vertex connectivity* of G, denoted by $\kappa(\overrightarrow{G})$, is the minimum number of vertices whose deletion yields the resulting digraph non-strongly connected. In this section, we characterize the digraph that minimizes the distance signless Laplacian spectral radius with given vertex connectivity.

Let $\mathcal{D}_{n,k}$ denote the set of strongly connected digraphs with order n and vertex connectivity $\kappa(\overrightarrow{G}) = k$. If $k = n-1$, then $\mathcal{D}_{n,n-1} = \{\overrightarrow{K_n}\}$. Thus, we only need to discuss the cases $1 \leq k \leq n-2$.

Let $\overrightarrow{G_1} \triangledown \overrightarrow{G_2}$ denote the digraph obtained from two disjoint digraphs $\overrightarrow{G_1}$, $\overrightarrow{G_2}$ with vertex set $V(\overrightarrow{G_1}) \cup V(\overrightarrow{G_2})$ and arc set $E = E(\overrightarrow{G_1}) \cup E(\overrightarrow{G_2}) \cup \{(u, v), (v, u) | u \in V(\overrightarrow{G_1}), v \in V(\overrightarrow{G_2})\}$. Let $1 \leq \kappa \leq n-2$, and $\overrightarrow{K}^{k,s}_{n-s}$ denote the digraphs $\overrightarrow{K}_k \triangledown (\overrightarrow{K}_s \cup \overrightarrow{K}_{n-k-s}) \cup E_1$, where $E_1 = \{(u, v) | u \in V(\overrightarrow{K}_s), v \in V(\overrightarrow{K}_{n-k-s})\}$. Let $\mathcal{K}_{n,k} = \{\overrightarrow{K}^{k,s}_{n-s} | 1 \leq s \leq n-s-1\}$. A digraph $\overrightarrow{G}$ is a *minimizing digraph* of $\mathcal{D}_{n,k}$ if $\overrightarrow{G} \in \mathcal{D}_{n,k}$ and $\eta(\overrightarrow{G}) = min\{\eta(\overrightarrow{G}) | \overrightarrow{G} \in \mathcal{D}_{n,k}\}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 438-446, December 2017.
http://repository.uwyo.edu/ela

Let $J_{a\times b}$ be the $a\times b$ matrix whose entries are all equal to 1, I_n be the $n\times n$ unit matrix and $A(\overrightarrow{G})$ be the arc set of digraph $\overrightarrow{G}$.

Lemma 4.1. [2] *Let $\overrightarrow{G}$ be an arbitrary strongly connected digraph with vertex connectivity k. Suppose that S is a k-vertex cut of $\overrightarrow{G}$ and $\overrightarrow{G_1},\dots,\overrightarrow{G_s}$ are the strongly connected components of $\overrightarrow{G}-S$. Then there exists an ordering of $\overrightarrow{G_1},\dots,\overrightarrow{G_s}$ such that, for $1\le i\le s$ and $v\in V(\overrightarrow{G_i})$, every tail of v in $\overrightarrow{G_1},\dots,\overrightarrow{G_i}$.*

Lemma 4.2. [8] *Let $\overrightarrow{G}$ be a strongly connected digraph with $u,v\in V(\overrightarrow{G})$ and $uv\notin A(\overrightarrow{G})$. Then $\eta(\overrightarrow{G})>\eta(\overrightarrow{G}+uv)$.*

By Lemma 4.1, we know that there exists a strongly connected component of $\overrightarrow{G}-S$, say $\overrightarrow{G_1}$, such that no vertex of $V(\overrightarrow{G_1})$ has in-neighbors in $\overrightarrow{G}-S-\overrightarrow{G_1}$. Let $\overrightarrow{G_1}'=\overrightarrow{G}-S-\overrightarrow{G_1}$. Next, we construct a new digraph $\overrightarrow{H}$ by adding to $\overrightarrow{G}$ any legal arcs from $\overrightarrow{G}[V(\overrightarrow{G_1})]$ to $\overrightarrow{G}[V(\overrightarrow{G_1})]\cup\overrightarrow{G}[V(\overrightarrow{G_1}')]$ or any legal arcs from $\overrightarrow{G}[V(\overrightarrow{G_1}')]$ to $\overrightarrow{G}[V(\overrightarrow{G_1}')]$ that were not present in $\overrightarrow{G}$. Obviously, $\overrightarrow{H}$ is also k-connected and $\overrightarrow{H}\in\overrightarrow{\mathcal{K}}^{k,s}_{n-s}$. Therefore, by Lemma 4.2, we have the digraphs which achieve the minimum distance signless Laplacian spectral radius among all digraphs in $\mathcal{D}_{n,k}$ must be in $\mathcal{K}_{n,k}$.

Theorem 4.3. *The digraph $\overrightarrow{\mathcal{K}}^{k,1}_{n-1}$ is the minimizing digraph among all digraphs in $\overrightarrow{\mathcal{K}}^{k,s}_{n-s}$. Furthermore, if $\overrightarrow{G}\in\mathcal{D}_{n,k}$, then $\eta(\overrightarrow{G})\ge\frac{3n}{2}-2+\frac{1+\sqrt{n^2+6n-8k-7}}{2}$ with equality if and only if $\overrightarrow{G}\cong\overrightarrow{\mathcal{K}}^{k,1}_{n-1}$.*

Proof. Let $\overrightarrow{G}$ be an arbitrary digraph in $\overrightarrow{\mathcal{K}}^{k,s}_{n-s}$ and S be a k-vertex cut of $\overrightarrow{G}$. Suppose that $\overrightarrow{G_1}$, $\overrightarrow{G_2}$ (with $|V(\overrightarrow{G_1})|=n_1$ and $|V(\overrightarrow{G_1})|=n_2=n-k-n_1$, respectively) are two strongly connected components of $\overrightarrow{G}-S$ and with arcs $E_2=\{u_1u_2\in E|u_1\in V(\overrightarrow{G_1}),u_2\in V(\overrightarrow{G_2})\}$. Clearly, $1\le n_1\le n-k-1$. Let $D^Q(\overrightarrow{G})$ be the distance signless Laplacian matrix of $\overrightarrow{G}$. Then

$$D^Q(\overrightarrow{G})=\begin{pmatrix} J_{n_1\times n_1}+(n-2)I_{n_1} & J_{n_1\times k} & J_{n_1\times n_2}\\ J_{k\times n_1} & J_{k\times k}+(n-2)I_k & J_{k\times n_2}\\ 2J_{n_2\times n_1} & J_{n_2\times k} & J_{n_2\times n_2}+(n+n_1-2)I_{n_1}\end{pmatrix}$$

and

$$P_{D^Q(\overrightarrow{G})}=|\lambda I_n-D^Q(\overrightarrow{G})|=(\lambda-n+2)^{n_1+k-2}(\lambda-n-n_1+2)^{n_2-1}f(\lambda),$$

where $f(\lambda)=\begin{vmatrix}\lambda-n-n_1+2 & -k & -n_2\\ -n_1 & \lambda-n-k+2 & -n_2\\ -2n_1 & -k & \lambda-n-n_1-n_2+2\end{vmatrix}$.

Thus,

$$\begin{aligned}P_{D^Q(\overrightarrow{G})}&=(\lambda-n+2)^{n_1+k-1}(\lambda-n-n_1+2)^{n_2-1}[\lambda^2+(-k-2n-2n_1-n_2+4)\lambda\\&\quad+n(k+n_1+n_2)+(k+n-n_2)n_1+n^2+n_1^2-4n-2k-4n_1-2n_2+4]\\&=(\lambda-n+2)^{n_1+k-1}(\lambda-n-n_1+2)^{n_2-1}[\lambda^2+(-3n-n_1+4)\lambda\\&\quad+2kn_1+2n^2+2n_1^2-6n-2n_1+4].\end{aligned}$$

Obviously,

$$\frac{3n}{2}-2+\frac{n_1+\sqrt{(k+n_2)^2+8n_1n_2}}{2}>n-2$$

and

$$\frac{3n}{2}-2+\frac{n_1+\sqrt{(k+n_2)^2+8n_1n_2}}{2}>n+n_1-2.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 438-446, December 2017.
http://repository.uwyo.edu/ela

Thus,

$$\eta(\overrightarrow{G}) = \frac{3n}{2} - 2 + \frac{n_1 + \sqrt{(k+n_2)^2 + 8n_1n_2}}{2}.$$

Since $n_2 = n - k - n_1$, we can assume that

$$\eta(\overrightarrow{G}) = f(n_1) = \frac{3n}{2} - 2 + \frac{n_1 + \sqrt{-7n_1^2 + (6n-8k)n_1 + n^2}}{2}$$

and

$$\frac{\partial^2(f(n_1))}{\partial(n_1)} = -\frac{(-8k+6n-14n_1)^2}{8(-8kn_1 + n^2 + 6nn_1 - 7n_1^2)^{\frac{3}{2}}} - \frac{7}{2\sqrt{-8kn_1 + n^2 + 6nn_1 - 7n_1^2}} < 0.$$

For fixed n and k, we can assume that $n > k+2$ since in case $n = k+2$ there is only one value of n_1 under consideration. Since $\frac{\partial^2(f(n_1))}{\partial(n_1)} < 0$, the function $f(n_1)$ is a convex function with respect to n_1. Combining $f(n_1) > 0$ for $1 \leq n_1 \leq n-k-1$, we know that the minimal value of $\eta(\overrightarrow{G})$ must be taken at either $n_1 = 1$ or at $n_1 = n-k-1$. Let $g(x) = -7x^2 + (6n-8k)x + n^2$, then $\beta = g(1) = (n+7)(n-1) - 8k > 0$ and $\alpha = g(n-k-1) = (k-3)^2 + 8(n-2) > 0$. Then

$$f(n-k-1) - f(1) = \frac{n-k-2+\sqrt{\alpha}-\sqrt{\beta}}{2} = \frac{1}{2}(n-k-2)\left(1 - \frac{n+k}{\sqrt{\alpha}+\sqrt{\beta}}\right).$$

If $f(n-k-1) - f(1) \leq 0$, then we can get that

$$\begin{cases} \sqrt{\alpha} + \sqrt{\beta} \leq n+k, \\ \sqrt{\alpha} - \sqrt{\beta} \leq -n+k+2. \end{cases}$$

By the above inequalities, we have $\alpha \leq (k+1)^2$ which reduces to $n \leq k+1$, a contradiction. Thus, $f(n-k-1) - f(1) > 0$, i.e., $\eta(\overrightarrow{\mathcal{K}}_{k+1}^{k,n-k-1}) > \eta(\overrightarrow{\mathcal{K}}_{n-1}^{k,1})$ and $\eta(\overrightarrow{G}) \geq \eta(\overrightarrow{\mathcal{K}}_{n-1}^{k,1}) = \frac{3n}{2} - 2 + \frac{1+\sqrt{n^2+6n-8k-7}}{2}$ with equality if and only if $\overrightarrow{G} \cong \overrightarrow{\mathcal{K}}_{n-1}^{k,1}$. ∎

Acknowledgment. The authors are grateful to the anonymous referees for their valuable comments and helpful suggestions, which have considerably improved the presentation of this paper. This work was supported by NSFC (no. 11531011 and no. 11461071) and NSFXJ (no. 2015KL019).

REFERENCES

[1] M. Aouchiche and P. Hansen. Two Laplacians for the distance matrix of a graph. *Linear Algebra and its Applications*, 439:21–33, 2013.

[2] J. Bondy and U. Murty. *Graph Theory with Applications.* Macmillan, London, 1976.

[3] S.S. Bose, M. Nath, and S. Paul. On the maximal distance spectral radius of graphs without a pendent vertex. *Linear Algebra and its Applications*, 438:4260–4278, 2013.

[4] Y. Chen, H. Lin, and J. Shu. Sharp upper bounds on the distance spectral radius of a graph. *Linear Algebra and its Applications*, 439:2659–2666, 2013.

[5] W. Hong and L. You. Some sharp bounds on the distance signless Laplacian spectral radius of graphs. *Mathematics*, 2013.

[6] A. Ilić. Distance spetral radius of trees with given matching number. *Discrete Applied Mathematics*, 158:1799–1806, 2010.

[7] G. Indulal. Sharp bounds on the distance spectral radius and the distance energy of graphs. *Linear Algebra and its Applications*, 430:106–113, 2009.

[8] D. Li, G.P. Wang, and J.X. Meng. Some results on the distance and distance signless Laplacian spectral radius of graphs and digraphs. *Applied Mathematics and Computation*, 293:218–225, 2017.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 438-446, December 2017.
http://repository.uwyo.edu/ela

[9] H. Lin and X. Lu. Bounds on the distance signless laplacian spectral radius in terms of clique number. *Linear Multilinear Algebra*, 63(9):1750–1759, 2015.

[10] H. Lin, W. Yang, H. Zhang, and J. Shu. The distance spectral radius of digraphs with given connectivity. *Discrete Mathematics*, 312:1849–1856, 2012.

[11] R. Xing, B. Zhou, and J. Li. On the distance signless laplacian spectral radius of graphs. *Linear Multilinear Algebra*, 62(10):1377–1387, 2014.

[12] R. Xing and B. Zhou. On the distance and distance signless Laplacian spectral radii of bicyclic graphs. *Linear Algebra and its Applications*, 439:3955–3963, 2013.