On the distance signless Laplacian spectral radius of graphs and digraphs

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ON THE DISTANCE SIGNLESS LAPLACIAN SPECTRAL RADIUS
OF GRAPHS AND DIGRAPHS

DAN LI†, GUOPING WANG‡, AND JIXIANG MENG†

Abstract. Let \( \eta(G) \) denote the distance signless Laplacian spectral radius of a connected graph \( G \). In this paper, bounds for the distance signless Laplacian spectral radius of connected graphs are given, and the extremal graph with the minimal distance signless Laplacian spectral radius among the graphs with given vertex connectivity and minimum degree is determined. Furthermore, the digraph that minimizes the distance signless Laplacian spectral radius with given vertex connectivity is characterized.

Key words. Distance signless Laplacian spectral radius, Bound, Extremal graph, Digraph, Vertex connectivity.

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1. Introduction. In this paper, we consider simple and connected graphs. Let \( G \) be a simple graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E(G) \). Let \( d_i \) be the degree of the vertex \( v_i \) in \( G \) for \( i = 1, \ldots, n \). Let \( A(G) = (a_{ij})_{n \times n} \) be the \((0,1)\)-adjacency matrix of \( G \), where \( a_{ij} = 1 \) if \( v_iv_j \in E(G) \) and \( a_{ij} = 0 \) otherwise. Let \( \Delta(G) = \text{diag}(d_1, \ldots, d_n) \) be the diagonal degree matrix. Then \( Q(G) = \Delta(G) + A(G) \) is the signless Laplacian matrix of \( G \).

Let \( D(G) = (d_{ij})_{n \times n} \) be the distance matrix of a connected graph \( G \), where \( d_{ij} = d_G(v_i, v_j) \) is defined to be the length of shortest path between \( v_i \) and \( v_j \). We call \( D_i = \sum_{j=1}^{n} d_{ij} \) the transmission of vertex \( v_i \) (\( i = 1, 2, \ldots, n \)). The matrix \( D^Q(G) = \text{Diag}(Tr) + D(G) \) is the distance signless Laplacian matrix of \( G \) (\( D^Q\)-matrix), where \( \text{Diag}(Tr) \) is the diagonal matrix whose \( i \)th diagonal entry is \( D_i \). The matrix \( D^Q(G) \) is nonnegative and irreducible when \( G \) is connected. The largest eigenvalue of \( D^Q(G) \) is called distance signless Laplacian spectral radius of the graph \( G \).

The distance spectral radius of a connected graph has been studied extensively. Chen, Lin and Shu [4] obtained sharp upper bounds on the distance spectral radius of a graph. Indulal [7] gave some bounds for the distance spectral radius of graphs. Ilić [6] obtained the tree with given matching number that minimizes the distance spectral radius. Bose, Nath and Paul [3] determined the unique graph with maximal distance spectral radius in the class of graphs without a pendent vertex. Lin, Yang, Zhang and Shu [10] characterized the extremal digraph with the minimal distance spectral radius among all digraphs with given vertex connectivity and the extremal graph with the minimal distance spectral radius among all graphs with given edge connectivity.

Aouchiche and Hansen [1] introduced the distance Laplacian and distance signless Laplacian spectra of graphs, respectively. Lin and Lu [9] found a sharp lower bound as well as a sharp upper bound of the distance signless Laplacian spectral radius in terms of the clique number. Furthermore, both extremal graphs are...
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uniquely determined. Hong and You [5] obtained some sharp bounds on the distance signless Laplacian spectral radius of graphs. Xing and Zhou [11] gave the unique graphs with minimum distance and distance signless Laplacian spectral radius among bicyclic graphs with fixed number of vertices. Xing, Zhou and Li [12] determined the graphs with minimum distance signless Laplacian spectral radius among the trees, unicyclic graphs, bipartite graphs, the connected graphs with fixed pendant vertices and fixed connectivity, respectively.

In this paper, we obtain the bounds for the distance signless Laplacian spectral radius of graphs and determine the extremal graph with the minimum distance signless Laplacian spectral radius among the graphs with given vertex connectivity and minimum degree. Furthermore, we characterize the digraph that minimizes the distance signless Laplacian spectral radius with given vertex connectivity.

2. Bounds on the distance signless Laplacian spectral radius of graphs. Let $G$ be a graph. We denote the distance signless Laplacian spectral radius of $G$ by $\eta(G)$. A graph $G$ is transmission regular if $D_1 = D_2 = \cdots = D_n$.

**Lemma 2.1.** [5] Let $G$ be a connected graph on $n$ vertices. Then
\[
\min\{D_i : 1 \leq i \leq n\} \leq \frac{\eta(G)}{2} \leq \max\{D_i : 1 \leq i \leq n\}.
\]
Moreover, one of the equalities holds if and only if $G$ is a transmission regular graph.

Let $G$ be a graph on $n$ vertices. It was shown in [11] that $\eta(G) \geq 2\frac{\sum_{i=1}^{n} D_i}{n}$ with equality if and only if $G$ is transmission regular.

Note that $\sum_{i=1}^{n} D_i^2 \geq \frac{1}{n}(\sum_{i=1}^{n} D_i)^2$. Then $2\sqrt{\frac{D_1^2 + D_2^2 + \cdots + D_n^2}{n}} \geq 2\frac{\sum_{i=1}^{n} D_i}{n}$. Thus, the following result gives a better lower bound of $\eta(G)$.

**Theorem 2.2.** Let $G$ be a graph on $n$ vertices. Then $\eta(G) \geq 2\sqrt{\frac{D_1^2 + D_2^2 + \cdots + D_n^2}{n}}$ with equality if and only if $G$ is transmission regular.

**Proof.** Let $x = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)$. Then $\eta(G) \geq \sqrt{x(D^Q(G))^2x^T}$.

Note that
\[
xD^Q(G) = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)D^Q(G) = \frac{2}{\sqrt{n}}(D_1, D_2, \ldots, D_n).
\]

We have
\[
x(D^Q(G))^2x^T = xD^Q(G)(xD^Q(G))^T = 4\frac{n}{n} \sum_{i=1}^{n} D_i^2.
\]

Thus, $\eta(G) \geq 2\sqrt{\frac{D_1^2 + D_2^2 + \cdots + D_n^2}{n}}$.

If $G$ is transmission regular, then by Lemma 2.1, $\eta(G) = 2D_i = 2\sqrt{\frac{1}{n} \sum_{i=1}^{n} D_i^2}$.

Conversely, if equality holds, then $x$ is the eigenvector corresponding to $\eta(G)$, i.e., $D^Q(G)x^T = \eta(G)x^T$. This implies that $D_i = \eta(G)/2$ ($i = 1, 2, \ldots, n$).
Let $\rho(G)$ be the distance spectral radius of the graph $G$. If $G$ is transmission regular, then $\rho(G) = D_i$. Thus, we have the following.

**Corollary 2.3.** Suppose that $G$ is transmission regular. Then $\eta(G) = 2\rho(G)$.

A graph $G$ is regular if $d_1 = \cdots = d_n$.

**Lemma 2.4.** Let $G$ be a connected graph with diameter $d \leq 2$. Then $G$ is regular if and only if $G$ is transmission regular.

**Proof.** If $d = 1$, then $G \cong K_n$. If $d = 2$, then $D_i = d_i + 2(n - 1 - d_i) = 2n - 2 - d_i$, from which we can see that $G$ is regular if and only if $G$ is transmission regular.

The unit eigenvector $x(G) = (x_1, x_2, \ldots, x_n)^T > 0$ corresponding to $\eta(G)$ is called the Perron vector of $D^Q(G)$.

**Theorem 2.5.** Let $G$ be a graph on $n$ vertices with maximum degree $\Delta_1$ and second maximum degree $\Delta_2$. Then $\eta(G) \geq 4n - 4 - \Delta_1 - \Delta_2$ with equality if and only if $G$ is a regular graph with diameter $d \leq 2$.

**Proof.** Let $x = (x_1, x_2, \ldots, x_n)^T$ be a Perron eigenvector of $D^Q(G)$ such that $x_i = \min_{1 \leq k \leq n} x_k$ and $x_j = \min_{1 \leq k \neq i \leq n} x_k$.

From $D^Q(G)x = \eta(G)x$, we obtain that

\[
\eta(G)x_i = D_ix_i + \sum_{k=1, k \neq i}^n d_{ik}x_k \geq D_i(x_i + x_j);
\]

\[
\eta(G)x_j = D_jx_j + \sum_{k=1, k \neq j}^n d_{jk}x_k \geq D_j(x_i + x_j).
\]

Note that

\[
D_i = \sum_{k=1, k \neq i}^n d_{ik} \geq d_i + (n - 1 - d_i)2 = 2n - 2 - d_i;
\]

\[
D_j = \sum_{k=1, k \neq j}^n d_{jk} \geq d_j + (n - 1 - d_j)2 = 2n - 2 - d_j.
\]

Then we can get that

\[
(2.1) \quad \eta(G)x_i \geq (2n - 2 - d_i)(x_i + x_j)
\]

and

\[
(2.2) \quad \eta(G)x_j \geq (2n - 2 - d_j)(x_i + x_j).
\]

Therefore, by (2.1) and (2.2), we have

\[
\eta(G)(x_i + x_j) \geq (4n - 4 - d_i - d_j)(x_i + x_j) \geq (4n - 4 - \Delta_1 - \Delta_2)(x_i + x_j),
\]

that is

\[
\eta(G) \geq 4n - 4 - \Delta_1 - \Delta_2.
\]

If the equality holds, then we deduce that $G$ is a regular graph with diameter $d \leq 2$ and all $x_i$ are equal. If $d = 1$, then $G \cong K_n$. If $d = 2$, then we get $\eta(G)x_i = 2(d_i + (n - 1 - d_i)2)x_i$, i.e., $\eta(G) = 2(2n - 2 - d_i)$, which implies that $G$ is a regular graph. Conversely, by Lemma 2.4, we can get that $\eta(G) = 2(2n - 2 - \Delta_1)$ if $G$ is a regular graph with diameter $d \leq 2$. \qed
Similarly, we can get that \( \eta(G) \leq 2dn - d(d - 1) - 2 - (d - 1)(\delta_1 + \delta_2) \) with equality if and only if \( G \) is a regular graph with diameter \( d \leq 2 \).

**Proof.** Let \( x = (x_1, x_2, \ldots, x_n)^T \) be a Perron eigenvector of \( D^Q(G) \) such that \( x_s = \max_{1 \leq k \leq n} x_k \) and \( x_t = \max_{1 \leq k \leq n} x_k \).

From \( D^Q(G)x = \eta(G)x \), we obtain that

\[
\eta(G)x_s = D_s x_s + \sum_{k=1, k \neq s}^{n} d_{sk} x_k \leq D_s (x_s + x_t);
\]

\[
\eta(G)x_t = D_t x_t + \sum_{k=1, k \neq t}^{n} d_{tk} x_k \leq D_t (x_s + x_t).
\]

Note that

\[
D_s = \sum_{k=1, k \neq s}^{n} d_{sk} \leq d_s + 2 + \cdots + d - 1 + d[n - 1 - d_s - (d - 2)] = dn - \frac{d(d - 1)}{2} - 1 - d_s(d - 1).
\]

Thus,

\[
(2.3) \quad \eta(G)x_s \leq \left[ dn - \frac{d(d - 1)}{2} - 1 - d_s(d - 1) \right] (x_s + x_t).
\]

Similarly, we can get that

\[
(2.4) \quad \eta(G)x_t \leq \left[ dn - \frac{d(d - 1)}{2} - 1 - d_t(d - 1) \right] (x_s + x_t).
\]

Therefore, by (2.3) and (2.4), we have

\[
\eta(G)(x_s + x_t) \leq \left[ 2dn - d(d - 1) - 2 - (d - 1)(\delta_1 + \delta_2) \right] (x_s + x_t),
\]

from which we obtain that \( \eta(G) \leq 2dn - d(d - 1) - 2 - (d - 1)(\delta_1 + \delta_2) \).

If the equality holds, then all \( x_i \) are equal, and hence, \( D^Q(G) \) has equal row sums. Therefore, \( G \) is transmission regular. If \( d \geq 3 \), then by (2.3), we know that for each vertex \( v_i \), there is exactly one vertex \( v_j \) such that \( d_{ij} = 2 \), and then \( d \leq 4 \). If the diameter of \( G \) is 3 and equality holds, then for a center vertex \( v_s \), from \( D^Q(G)x = \eta(G)x \) and (2.3), written for the component \( x_s \), we have

\[
\eta(G)x_s = D_s x_s + d_s x_s + (n - 1 - d_s)2x_s = 2(2n - 2 - d_s)x_s = 2(3n - 4 - 2d_s)x_s,
\]

and thus, \( d_s = n - 2 \), which implies that \( G \cong P_4 \), but \( P_4 \) is not transmission regular, a contradiction. Therefore, \( G \) is a regular graph with diameter \( d \leq 2 \). Conversely, by Lemma 2.4, we can get that \( \eta(G) = 2(2n - 2 - \delta_1) \) if \( G \) is a regular graph with diameter \( d \leq 2 \).

3. The minimal distance signless Laplacian spectral radius for graphs with given vertex connectivity and minimum degree. The vertex connectivity of a graph \( G \) is the minimum number of vertices, whose deletion results in a disconnected or a trivial graph. The join \( G_1 + G_2 \) of two graphs \( G_1(V_1, E_1) \) and \( G_2(V_2, E_2) \) consists of the union \( G_1 \cup G_2 \) of \( G_1 \) and \( G_2 \) and all edges joining \( V_1 \) with \( V_2 \). Let
\( G_{k,n} \) be the class of all graphs of order \( n \) with vertex connectivity \( \kappa(G) \leq k \leq n-1 \) and minimum degree \( \delta \geq k \). Let \( G_{k,\delta,n} = K_k + (K_{\delta-k+1} \cup K_{n-\delta-1}) \). Obviously, \( G_{k,\delta,n} \in G_{k,n} \).

For fixed \( n \) and \( k \), let
\[
g(n_1, n_2) = x^3 - a_1 x^2 - a_2 x + a_3,
\]
where \( a_1 = 4n_1 + n_1 + n_2 - 6 \), \( a_2 = (n_1 + n_2)(20 - 13k) - 8(n_1^2 + n_2^2) - 12(n_1n_2 + 1) + 5k^2 + 16k \), and \( a_3 = -2k^3 + (10 - 8n_2 - 8n_1)k^2 + (-16 + 26n_2 + 26n_1 - 10n_2^2 - 16n_1n_2)k - 4(n_2 - 2)^2 + (n_1 - 2)^2 + (n_1 + n_2)(2n_1n_2 + 1) + 2(3n_1n_2 + 1) \).

**Lemma 3.1.** \( \eta = \eta(K_k + (K_{n_1} \cup K_{n_2})) \) satisfies the equation
\[
g(n_1, n_2) = 0.
\]

**Proof.** Let \( \mathbf{x} \) be the Perron vector of \( D^Q(K_k + (K_{n_1} \cup K_{n_2})) \). It is clear that \( \mathbf{x} \) can be written as
\[
\mathbf{x} = \left( \begin{array}{c} x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2} \end{array} \right)^T.
\]
From \( D^Q(K_k + (K_{n_1} \cup K_{n_2})) \mathbf{x} = \eta \mathbf{x} \), we have
\[
\eta x = (2n_1 + 2n_2 + k - 2)x + ky + 2n_2z;
\]
\[
\eta y = n_1x + (n_1 + n_2 + 2k - 2)y + nz;
\]
\[
\eta z = 2n_1x + ky + (2n_1 + 2n_2 + k - 2)z.
\]
This shows that \( \eta \) is an eigenvalue of the matrix \( A \), where
\[
A = \begin{pmatrix}
2n_1 + 2n_2 + k - 2 & k & 2n_2 \\
2n_1 + 2n_2 + k - 2 & k & 2n_2 \\
2n_1 + 2n_2 + k - 2 & 2n_1 + 2n_2 + k - 2 & 2n_1 + 2n_2 + k - 2
\end{pmatrix},
\]
and \( \det(Ix - A) = g(n_1, n_2) = x^3 - a_1 x^2 - a_2 x + a_3 \). This shows that the result is true.

By Lemma 3.1 we can immediately get the following corollary.

**Corollary 3.2.** Spectral radius of \( G_{k,\delta,n} \) is the largest root of the equation
\[
T^3 - (5n - k - 6)T^2 - (-4k^2 + (4k + 4n - 8)\delta - nk - 8n^2 + 24n - 16)x + ((-12k + 8n + 16)\delta^2 + (12k^2 + 4kn - 8n^2 - 40k + 32)\delta - 4k^3 - 2kn^2 - 8k^2 + 54kn - 12n^2 - 56k + 4n - 24) = 0.
\]

**Lemma 3.3.** If \( n_2 \geq n_1 \geq 2 \), then \( \eta(K_k + (K_{n_1} \cup K_{n_2})) > \eta(K_k + (K_{n_1-1} \cup K_{n_2+1})) \).

**Proof.** Let
\[
h(\lambda) = g_{n_1,n_2}(\lambda) - g_{n_1-1,n_2+1}(\lambda) = -4(n_2 - n_1 + 1)\lambda + 4(k(n_2 - n_1 + 1) + (n_2 - n_1)(n_2 + n_1 - 1) + 2(n_1 - 1)).
\]
Then \( \lambda_0 = \frac{k + \frac{(n_2 - n_1)(n_2 + n_1 - 1)}{n_2 - n_1 + 1} + \frac{2(n_1 - 1)}{n_2 - n_1 + 1}}{n_2 - n_1 + 1} \) is the root of the equation \( h(\lambda) = 0 \).

By Lemma 2.1, we know that \( \eta(K_k + (K_{n_1} \cup K_{n_2})) \geq 2(n_2 + n_1 - 1) + 2k \). And,
\[
2(n_2 + n_1 - 1 + k) - \lambda_0 = 2(n_2 + n_1 - 1) + k - \frac{(n_2 - n_1)(n_2 + n_1 - 1)}{n_2 - n_1 + 1} - \frac{2(n_1 - 1)}{n_2 - n_1 + 1}
\]
\[
= 1 - \frac{n_2 - n_1}{n_2 - n_1 + 1} (n_2 + n_1 - 1) + \frac{(n_1 - 1)(n_2 - n_1 + 1) - 2n_1 + 2}{n_2 - n_1 + 1} > 0.
\]
Thus, \( \eta(K_k + (K_{n_1} \cup K_{n_2})) > \lambda_0 \) and \( \eta(K_k + (K_{n_1-1} \cup K_{n_2+1})) > \lambda_0 \). Since \( h(\lambda) \) is a decreasing function of \( \lambda \), \( g_{n_1,n_2}(\lambda) - g_{n_1-1,n_2+1}(\lambda) < 0 \) for \( \lambda > \lambda_0 \), which yields \( \eta(K_k + (K_{n_1} \cup K_{n_2})) > \eta(K_k + (K_{n_1-1} \cup K_{n_2+1})) \).


**Lemma 3.4.** [11] Let $G$ be a connected graph with $u, v \in V(G)$ and $uv \notin E(G)$. Then $\eta(G) > \eta(G + uv)$.

**Theorem 3.5.** Among all the connected graphs of order $n$ with connectivity at most $k$ and minimum degree $\delta$, $G_{k, \delta, n}$ uniquely minimizes the distance signless Laplacian spectral radius.

**Proof.** The case $k = n - 1$ is trivial, and so we assume that $k \leq n - 2$. Let $G$ be a graph in $G_{k, n}$ with the minimum distance signless Laplacian spectral radius, and $S$ be a $k$-vertex cut of $G$. Now we verify the following two claims.

**Claim 1.** $G - S$ consists exactly of two components.

Suppose to the contrary that $G - S$ contains $G_1$, $G_2$ and $G_3$. Then there must be $u \in G_2$ and $v \in G_3$ such that $uv \notin E(G)$. Thus, by Lemma 3.4, $\eta(G) > \eta(G + uv)$ while $G + uv \in G_{k, n}$. This contradiction shows that $G - S = G_1 \cup G_2$.

**Claim 2.** $G_1$ and $G[V(G_i) \cup S]$ $(i = 1, 2)$ are all complete graphs.

If there is $uv \notin E(G_i)$, then $G + uv \in G_{k, n}$. By Lemma 3.4, $\eta(G) > \eta(G + uv)$, a contradiction. Similarly, $G[V(G_i) \cup S]$ is also a complete graphs.

By the above argument we can determine that $G \cong K_k + (K_{n_1} \cup K_{n_2})$. Since $\delta \geq k$, we have $n_i \geq \delta - k + 1$. If $n_1 \geq \delta - k + 2$, without loss of generality, we suppose $n_2 \geq n_1$, then $\eta(G) > \eta(K_k + (K_{n_1} - 1) \cup K_{n_2} + 1))$ while $K_k + (K_{n_1} - 1 \cup K_{n_2} + 1) \in G_{k, n}$ by Lemma 3.3. This contradiction shows that $n_1 = \delta - k + 1$ and $n_2 = n - 1$, that is, $G \cong G_{k, \delta, n}$.

Let $C(n, k)$ be the set of connected graphs with $n$ vertices and connectivity $k$, where $1 \leq k \leq n - 2$, then the following result can be obtained immediately.

**Corollary 3.6.** [12] Let $G \in C(n, k)$, where $1 \leq k \leq n - 2$. Then, $\eta(G) \geq \eta(K_k + (K_1 \cup K_{n-k-1}))$ with equality if and only if $G = K_k + (K_1 \cup K_{n-k-1})$.

4. The minimal distance signless Laplacian spectral radius for digraphs with given vertex connectivity. Similar to undirected graph, we use $D(G) = (d_{ij})_{n \times n}$ and $D^Q(G) = \text{Diag}(Tr) + D(G)$ to denote the distance matrix and distance signless Laplacian matrix of $G$, respectively, where $d_{ij} = d_{ij}(v_i, v_j)$ denote the length of shortest dipath from $v_i$ to $v_j$ in $G$ and $\text{Diag}(Tr)$ is the diagonal matrix with $D_i$. Clearly, the matrix $D^Q(G)$ is irreducible when $G$ is strongly connected. The eigenvalue of $D^Q(G)$ with the largest modulus is the distance signless Laplacian spectral radius of $G$, denoted by $\eta(G)$. For a strongly connected digraph $\vec{G} = (V(\vec{G}), E(\vec{G}))$, a set of vertices $S \subseteq V(\vec{G})$ is a vertex cut if $\vec{G} - S$ is not strongly connected. The vertex connectivity of $G$, denoted by $\kappa(G)$, is the minimum number of vertices whose deletion yields the resulting digraph non-strongly connected. In this section, we characterize the digraph that minimizes the distance signless Laplacian spectral radius with given vertex connectivity.

Let $D_{n,k}$ denote the set of strongly connected digraphs with order $n$ and vertex connectivity $\kappa(\vec{G}) = k$. If $k = n - 1$, then $D_{n,n-1} = \{\vec{K}_n\}$. Thus, we only need to discuss the cases $1 \leq k \leq n - 2$.

Let $\vec{G}_1 \cup \vec{G}_2$ denote the digraph obtained from two disjoint digraphs $\vec{G}_1$, $\vec{G}_2$ with vertex set $V(\vec{G}_1) \cup V(\vec{G}_2)$ and arc set $E = E(\vec{G}_1) \cup E(\vec{G}_2) \cup \{(u, v), (v, u) | u \in V(\vec{G}_1), v \in V(\vec{G}_2)\}$. Let $1 \leq \kappa \leq n - 2$, and $\vec{K}_{n-s}$ denote the digraphs $\vec{K}_n \cap (\vec{K}_s \cup \vec{K}_{n-k-s}) \cup E_1$, where $E_1 = \{(u, v) | u \in V(\vec{K}_s), v \in V(\vec{K}_{n-k-s})\}$. Let $\vec{K}_{n,k} = \{\vec{K}_{n,s} | 1 \leq s \leq n - s - 1\}$. A digraph $\vec{G}$ is a minimizing digraph of $D_{n,k}$ if $\vec{G} \in D_{n,k}$ and $\eta(\vec{G}) = \min \{\eta(\vec{G}) | \vec{G} \in D_{n,k}\}$.
Let \( J_{a \times b} \) be the \( a \times b \) matrix whose entries are all equal to 1, \( I_n \) be the \( n \times n \) unit matrix and \( A(G) \) be the arc set of digraph \( G \).

**Lemma 4.1.** [2] Let \( G \) be an arbitrary strongly connected digraph with vertex connectivity \( k \). Suppose that \( S \) is a \( k \)-vertex cut of \( G \) and \( G_1, \ldots, G_s \) are the strongly connected components of \( G - S \). Then there exists an ordering of \( G_1, \ldots, G_s \) such that, for \( 1 \leq i \leq s \) and \( v \in V(G_i) \), every tail of \( v \) in \( G_1, \ldots, G_i \).

**Lemma 4.2.** [8] Let \( G \) be a strongly connected digraph with \( u,v \in V(G) \) and \( uv \notin A(G) \). Then \( \eta(G) > \eta(G + uv) \).

By Lemma 4.1, we know that there exists a strongly connected component of \( G - S \), say \( G_1 \), such that no vertex of \( V(G_1) \) has in-neighbors in \( G - S - G_1 \). Let \( G_1' = G - S - G_1 \). Next, we construct a new digraph \( \overline{H} \) by adding to \( G \) any legal arcs from \( \overline{G}[V(G_1')] \) to \( \overline{G}[V(G_1')] \cup \overline{G}[V(G_1')] \) or any legal arcs from \( \overline{G}[V(G_1')] \) to \( \overline{G}[V(G_1')] \) that were not present in \( G \). Obviously, \( \overline{H} \) is also \( k \)-connected and \( \overline{H} \in K_{k-1}^{n,k} \).

Therefore, by Lemma 4.2, we have the digraphs which achieve the minimum distance signless Laplacian spectral radius among all digraphs in \( D_{n,k} \) must be in \( K_{n,k} \).

**Theorem 4.3.** The digraph \( K_{n,k}^{n,k} \) is the minimizing digraph among all digraphs in \( K_{n,k} \). Furthermore, if \( \overline{G} \in D_{n,k} \), then \( \eta(\overline{G}) \geq \frac{3n}{2} - 2 + \frac{1+\sqrt{4n^2+8n^3-8k-7}}{2} \) with equality if and only if \( \overline{G} \equiv K_{n,k}^{n,k} \).

**Proof.** Let \( \overline{G} \) be an arbitrary digraph in \( K_{n,k}^{n,k} \) and \( S \) be a \( k \)-vertex cut of \( \overline{G} \). Suppose that \( \overline{G}_1, \overline{G}_2 \) (with \( |V(\overline{G}_1)| = n_1 \) and \( |V(\overline{G}_2)| = n_2 = n - k - n_1 \), respectively) are two strongly connected components of \( \overline{G} - S \) and with arcs \( E_2 = \{u_1u_2 \in E | u_1 \in V(\overline{G}_1), u_2 \in V(\overline{G}_2) \} \). Clearly, \( 1 \leq n_1 \leq n - k - 1 \). Let \( D^Q(\overline{G}) \) be the distance signless Laplacian matrix of \( \overline{G} \). Then

\[
D^Q(\overline{G}) = \begin{pmatrix}
J_{n_1 \times n_1} + (n - 2)I_{n_1} & J_{n_1 \times k} & J_{n_1 \times n_2} \\
J_{k \times n_1} & J_{k \times k} + (n - 2)I_{k} & J_{k \times n_2} \\
2J_{n_2 \times n_1} & 2J_{n_2 \times k} & J_{n_2 \times n_2} + (n + n_1 - 2)I_{n_1}
\end{pmatrix}
\]

and

\[
P_{D^Q(\overline{G})} = \lambda I_n - D^Q(\overline{G}) = (\lambda - n + 2)^{n_1+k-2}(\lambda - n - n_1 + 2)^{n_2-1}f(\lambda),
\]

where

\[
f(\lambda) = \begin{vmatrix}
\lambda - n - n_1 + 2 & -k & -n_2 \\
-n_1 & \lambda - n - k + 2 & -n_2 \\
-2n_1 & -k & \lambda - n - n_1 - n_2 + 2
\end{vmatrix}.
\]

Thus,

\[
P_{D^Q(\overline{G})} = (\lambda - n + 2)^{n_1+k-1}(\lambda - n - n_1 + 2)^{n_2-1}[\lambda^2 + (-k - 2n - 2n_1 - n_2 + 4)\lambda + n(k + n_1 + n_2) + (k + n_1 - n_2)n_1 + n_1^2 - 4n - 2k - 4n_1 - 2n_2 + 4]
\]

\[= (\lambda - n + 2)^{n_1+k-1}(\lambda - n - n_1 + 2)^{n_2-1}[\lambda^2 + (-3n - n_1 + 4)\lambda + 2kn_1 + 2n_1^2 + 2n_2^2 - 6n - 2n_1 + 4].\]

Obviously,

\[
\frac{3n}{2} - 2 + \frac{n_1 + \sqrt{(k + n_2)^2 + 8n_1n_2}}{2} > n - 2
\]

and

\[
\frac{3n}{2} - 2 + \frac{n_1 + \sqrt{(k + n_2)^2 + 8n_1n_2}}{2} > n + n_1 - 2.
\]
Thus, 
\[ \eta(G) = \frac{3n}{2} - 2 + \frac{n_1 + \sqrt{(k + n_2)^2 + 8n_1n_2}}{2}. \]
Since \( n_2 = n - k - 1 \), we can assume that
\[ \eta(G) = f(n_1) = \frac{3n}{2} - 2 + \frac{n_1 + \sqrt{-7n_1^2 + (6n - 8k)n_1 + n^2}}{2} \]
and
\[ \frac{\partial^2 f(n_1)}{\partial n_1^2} = -\frac{(-8k + 6n - 14n_1)^2}{8(-8kn_1 + n^2 + 6n_1 - 7n_1^2)^2} - \frac{7}{2\sqrt{-8kn_1 + n^2 + 6n_1 - 7n_1^2}} < 0. \]

For fixed \( n \) and \( k \), we can assume that \( n > k + 2 \) since in case \( n = k + 2 \) there is only one value of \( n_1 \) under consideration. Since \( \frac{\partial^2 f(n_1)}{\partial n_1^2} < 0 \), the function \( f(n_1) \) is a convex function with respect to \( n_1 \). Combining \( f(n_1) > 0 \) for \( 1 \leq n_1 \leq n - k - 1 \), we know that the minimal value of \( \eta(G) \) must be taken at either \( n_1 = 1 \) or \( n_1 = n - k - 1 \). Let \( g(x) = -7x^2 + (6n - 8k)x + n^2 \), then \( \beta = g(1) = (n + 7)(n - 1) - 8k > 0 \) and \( \alpha = g(n - k - 1) = (k - 3)^2 + 8(n - 2) > 0 \). Then
\[ f(n - k - 1) - f(1) = \frac{n - k - 2 + \sqrt{\alpha} - \sqrt{\beta}}{2} = \frac{1}{2}(n - k - 2) \left( 1 - \frac{n+k}{\sqrt{\alpha} + \sqrt{\beta}} \right). \]
If \( f(n - k - 1) - f(1) \leq 0 \), then we can get that
\[ \left\{ \begin{array}{l} \sqrt{\alpha} + \sqrt{\beta} \leq n + k, \\ \sqrt{\alpha} - \sqrt{\beta} \leq -n + k + 2. \end{array} \right. \]
By the above inequalities, we have \( \alpha \leq (k + 1)^2 \) which reduces to \( n \leq k + 1 \), a contradiction. Thus, \( f(n - k - 1) - f(1) > 0 \), i.e., \( \eta(K_{k,n-k-1}) > \eta(K_{k,n-k+1}) \) and \( \eta(G) = \eta(K_{k,n-k+1}) = \frac{3n}{2} - 2 + \frac{1 + \sqrt{n^2 + 6n - 8k}}{2} \) with equality if and only if \( G \cong K_{k,n-k+1} \).

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