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Qi Kong

Northwestern Polytechnical University, kongqixgd@163.com

Ligong Wang

Northwestern Polytechnical University, lgwangmath@163.com

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UPPER BOUNDS ON THE Q-SPECTRAL RADIUS OF BOOK-FREE AND/OR $K_{S,T}$ -FREE GRAPHS*

QI KONG[†] AND LIGONG WANG[†]

Abstract. In this paper, two results about the signless Laplacian spectral radius $q(G)$ of a graph G of order n with maximum degree Δ are proved. Let $B_n = K_2 + \overline{K_n}$ denote a book, i.e., the graph B_n consists of n triangles sharing an edge. The results are the following:

(1) Let $1 < k \leq l < \Delta < n$ and G be a connected $\{B_{k+1}, K_{2,l+1}\}$ -free graph of order n with maximum degree Δ . Then

$$q(G) \leq \frac{1}{4} \left[3\Delta + k - 2l + 1 + \sqrt{(3\Delta + k - 2l + 1)^2 + 16l(\Delta + n - 1)} \right]$$

with equality if and only if G is a strongly regular graph with parameters (Δ, k, l) .

(2) Let $s \geq t \geq 3$, and let G be a connected $K_{s,t}$ -free graph of order n ($n \geq s + t$). Then

$$q(G) \leq n + (s - t + 1)^{1/t} n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3.$$

Key words. Complete bipartite subgraph, Zarankiewicz problem, Signless Laplacian spectral radius.

AMS subject classifications. 05C50, 15A18.

1. Introduction. Our graph notation follows Bollobás [1]. In particular, let $G = (V(G), E(G))$ be a simple graph. Denote by $v(G)$ the order of G and $e(G)$ the size of G , that is to say, $v(G) = |V(G)|$, and $e(G) = |E(G)|$. Set $\Gamma_G(u) = \{v | uv \in E(G)\}$, and $d_G(u) = |\Gamma_G(u)|$, or simply $\Gamma(u)$ and $d(u)$, respectively. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote the minimal degree and maximal degree of graph G , respectively.

For a simple graph G of order n , let $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$, and $A(G) = (a_{ij})_{n \times n}$ be the adjacency matrix of G with $a_{ij} = 1$ if v_i is adjacent to v_j , and $a_{ij} = 0$ otherwise. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G . The largest eigenvalue of $A(G)$ and $Q(G)$ are called spectral radius and signless Laplacian spectral radius (or Q-spectral radius) of G and denoted by $\rho(G)$ and $q(G)$, respectively.

Let X be a set of vertices of G . Then $G[X]$ is the graph induced by X , and $e(X) = e(G[X])$. Let P_k , C_k and K_k be the path, cycle, and complete graph of order k , respectively. If all vertices of G have the same degree k , then G is k -regular. A k -regular graph is called *strongly regular* with parameters (k, a, c) whenever each pair of adjacent vertices have $a \geq 0$ common neighbors, and each pair of non-adjacent vertices have $c \geq 1$ common neighbors.

The main results of this paper are in the spirit of the trend in the famous Zarankiewicz problem [9]:

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[†]Department of Applied Mathematics, School of Science, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, P.R. China (lgwangmath@163.com, kongqixgd@163.com). This work was supported by the National Natural Science Foundation of China (no. 11171273) and the Seed Foundation of Innovation and Creation for Graduate Students in Northwestern Polytechnical University (no. Z2016170).

PROBLEM A. How many edges can a graph of order n have if it does not contain a complete bipartite subgraph $K_{s,t}$?

In 1996, Füredi [4] gave an upper bound on the above Zarankiewicz problem. In 2010, Nikiforov [6] improved his result. That is, if G is a $K_{s,t}$ -free graph of order n , then

$$e(G) \leq \frac{1}{2}(s-t+1)^{1/t}n^{2-1/t} + \frac{1}{2}(t-1)n^{2-2/t} + \frac{1}{2}(t-2)n.$$

The spectral version of the Zarankiewicz problem is the following one:

PROBLEM B. How large can be the spectral radius $\rho(G)$ of a graph G of order n that does not contain $K_{s,t}$?

There are some results for some value of s and t .

In 2007, the upper bound on the signless Laplacian spectral radius of $K_{2,l+1}$ -free graph as the corollary of the following Lemma 1.1 was proved in [9] by Shi and Song.

LEMMA 1.1. Let $0 \leq k \leq l \leq \Delta < n$ and G be a connected $\{B_{k+1}, K_{2,l+1}\}$ -free graph of order n with maximum degree Δ . Then

$$\rho(G) \leq \frac{1}{2} \left[k - l + \sqrt{(k-l)^2 + 4\Delta + 4l(n-l)} \right]$$

with equality if and only if G is a strongly regular with parameters (Δ, k, l) .

In 2007, Nikiforov [7] improved the above bound showing that:

LEMMA 1.2. Let $l \geq k \geq 0$. If G is a $\{B_{k+1}, K_{2,l+1}\}$ -free graph of order n with maximum degree Δ . Then

$$\rho(G) \leq \min \left\{ \Delta, \frac{1}{2} \left[k - 1 + 1 + \sqrt{(k-l+1)^2 + 4l(n-1)} \right] \right\}.$$

If G is connected, equality holds if and only if one of the following conditions holds:

- (1) $\Delta^2 - \Delta(k-l+1) \leq l(n-1)$ and G is Δ -regular;
- (2) $\Delta^2 - \Delta(k-l+1) > l(n-1)$ and every two vertices of G have k common neighbors if they are adjacent, and l common neighbors, otherwise.

Setting $l = \Delta$ or $k = l$, Lemma 1.2 strengthens Corollaries 1 and 2 of [8].

In 2010, Nikiforov [6] also gave a bound as the following lemma.

LEMMA 1.3. Let $s \geq t \geq 2$, and let G be a $K_{s,t}$ -free graph of order n . If $t = 2$, then

$$\rho(G) \leq \frac{1}{2} + \sqrt{(s-1)(n-1) + 1/4}.$$

If $t \geq 3$, then

$$\rho(G) \leq (s-t+1)^{1/t}n^{1-1/t} + (t-1)n^{1-2/t} + t - 2$$

and

$$e(G) < \frac{1}{2}(s-t+1)^{1/t}n^{2-1/t} + \frac{1}{2}(t-1)n^{2-2/t} + \frac{1}{2}(t-2)n.$$

A newer trend in extremal graph theory is the Zarankiewicz problem for the signless Laplacian spectral radius of graphs:

PROBLEM C How large can the signless Laplacian spectral radius of a graph of order G be, if it does not contain $K_{s,t}$ as a subgraph?

When $s = t = 2$, we notice that the $K_{2,2}$ -free graph is the same as C_4 -free graph. Also in 2013, de Freitas et al. [2] have proved that if G contains no C_4 , then

$$q(G) < q(F_n),$$

unless $G = F_n$, where F_n is the friendship graph of order n . For n odd, F_n is a union of $\lfloor n/2 \rfloor$ triangles sharing a single common vertex, and for n even, F_n is obtained by hanging an edge to the common vertex of F_{n-1} .

In Section 2, we will prove the following results which give upper bounds on the signless Laplacian spectral radius of Book-free and/or $K_{2,l+1}$ -free ($l > 1$) graphs of order n with maximum degree Δ .

THEOREM 1.4. *Let $1 < k \leq l < \Delta < n$ and G be a connected $\{B_{k+1}, K_{2,l+1}\}$ -free graph of order n with maximum degree Δ . Then*

$$(1.1) \quad q(G) \leq \frac{1}{4} \left[3\Delta + k - 2l + 1 + \sqrt{(3\Delta + k - 2l + 1)^2 + 16l(\Delta + n - 1)} \right]$$

with equality if and only if G is a strongly regular graph with parameters (Δ, k, l) .

Because every graph is obviously $K_{2,\Delta+1}$ -free, Theorem 1.4 readily implies a sharp upper bound for book-free graph.

COROLLARY 1.5. *Let $1 < k < \Delta < n$ and G be a connected B_{k+1} -free graph of order n with maximum degree Δ . Then*

$$q(G) \leq \frac{1}{4} \left[\Delta + k + 1 + \sqrt{(\Delta + k + 1)^2 + 32\Delta(n - 1)} \right]$$

with equality if and only if G is a strongly regular graph with parameters (Δ, k, Δ) .

Because a $K_{2,l}$ -free graph is also B_l -free. Theorem 1.4 with $k = l$ also implies a sharp upper bound for $K_{2,l}$ -free graphs.

COROLLARY 1.6. *Let $1 < l < \Delta$ and G be a connected $K_{2,l+1}$ -free graph of order n with maximum degree Δ . Then*

$$q(G) \leq \frac{1}{4} \left[3\Delta - l + 1 + \sqrt{(3\Delta - l + 1)^2 + 32l(n - 1)} \right]$$

with equality if and only if G is a strongly regular graph with parameters (Δ, l, l) .

Furthermore, we will discuss $s \geq t \geq 3$. Let G be a connected graph of order n . Since G contains no $K_{s,t}$ when $n < s + t$, we only discuss the case $n \geq s + t$.

THEOREM 1.7. *Let $s \geq t \geq 3$, and let G be a connected $K_{s,t}$ -free graph of order n ($n \geq s + t$). Then*

$$q(G) \leq n + (s - t + 1)^{1/t} n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3.$$

2. Some known lemmas. In this section, we state two known results that will be used in this paper.

LEMMA 2.1. *Let $s \geq 2$, $t \geq 2$, $0 \leq k \leq s - 2$, and let $G(A, B)$ be a bipartite graph with parts A and B . Suppose that G contains no copy of $K_{s,t}$ with a vertex class of size s in A and a vertex class of size t in B . Then $G(A, B)$ has at most*

$$(s - k - 1)^{1/t}|B||A|^{1-1/t} + (t - 1)|A|^{1+k/t} + k|B|$$

edges.

LEMMA 2.2. ([3, 5]) *For every graph G , we have*

$$q(G) \leq \max_{u \in V(G)} \left\{ d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v) \right\}.$$

3. Proofs.

Proof of Theorem 1.4. Let Q_i denote the i th row vector of $Q = Q(G)$ and let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ be the Perron-eigenvector of Q corresponding to $q(G)$. Then $x_i > 0$ for $1 \leq i \leq n$. Since G is $\{B_{k+1}, K_{2,l+1}\}$ -free, each pair of adjacent vertices has at most k common neighbors and each pair of non-adjacent vertices has at most l common neighbors. Thus,

$$(3.2) \quad \sum_{i=1}^n \sum_{v_p, v_q \in \Gamma(v_i)} x_p x_q \leq k \sum_{v_p v_q \in E(G)} x_p x_q + l \sum_{v_p v_q \notin E(G)} x_p x_q.$$

Note that $\mathbf{x}^T A(K_n) \mathbf{x} \leq \rho(K_n) = n - 1$. Thus,

$$\begin{aligned} q(G) &= \mathbf{x}^T Q \mathbf{x} = \mathbf{x}^T D \mathbf{x} + \mathbf{x}^T A \mathbf{x} = \sum_{i=1}^n d_i x_i^2 + 2 \sum_{v_i v_p \in E(G)} x_i x_p \\ &\leq \Delta + \mathbf{x}^T A(K_n) \mathbf{x} - 2 \sum_{v_i v_p \notin E(G)} x_i x_p \\ &\leq \Delta + n - 1 - 2 \sum_{v_i v_p \notin E(G)} x_i x_p. \end{aligned}$$

Also we can obtain

$$\begin{aligned} q(G) &= \mathbf{x}^T Q \mathbf{x} = \sum_{i=1}^n \sum_{j=1, i < j}^n 2q_{i,j} x_i x_j + \sum_{i=1}^n d_i x_i^2 \\ &\leq \sum_{i=1}^n \sum_{j=1, i < j}^n q_{i,j} (x_i^2 + x_j^2) + \sum_{i=1}^n d_i x_i^2 \\ &= \sum_{i=1}^n \sum_{j=1, i < j}^n q_{i,j} x_i^2 + \sum_{i=1}^n d_i x_i^2 \\ &= 2 \sum_{i=1}^n d_i x_i^2. \end{aligned}$$

So

$$\sum_{i=1}^n d_i x_i^2 \geq \frac{q}{2}.$$

Then

$$\begin{aligned}
 q^2(G) &= \|Q\mathbf{x}\|^2 = \sum_{i=1}^n (Q_i\mathbf{x})^2 = \sum_{i=1}^n \left(d_i x_i + \sum_{v_i v_p \in E(G)} x_p \right)^2 \\
 &= \sum_{i=1}^n \left[d_i^2 x_i^2 + 2d_i x_i \sum_{v_i v_p \in E(G)} x_p + \left(\sum_{v_i v_p \in E(G)} x_p \right)^2 \right] \\
 &= \sum_{i=1}^n d_i^2 x_i^2 + 2 \sum_{i=1}^n d_i \sum_{v_i v_p \in E(G)} x_i x_p + \sum_{i=1}^n d_i x_i^2 + 2 \sum_{i=1}^n \sum_{v_p, v_q \in \Gamma(v_i)} x_p x_q \\
 (3.3) \quad &\leq (\Delta + 1) \sum_{i=1}^n d_i x_i^2 + 2\Delta \sum_{i=1}^n \sum_{v_i v_p \in E(G)} x_i x_p + 2k \sum_{v_p v_q \in E(G)} x_p x_q + 2l \sum_{v_p v_q \notin E(G)} x_p x_q \\
 &= (\Delta + 1) \sum_{i=1}^n d_i x_i^2 + (4\Delta + 2k) \sum_{v_i v_p \in E(G)} x_i x_p + 2l \sum_{v_p v_q \notin E(G)} x_p x_q \\
 &\leq (2\Delta + k) \left(\sum_{i=1}^n d_i x_i^2 + 2 \sum_{v_i v_p \in E(G)} x_i x_p \right) \\
 &\quad (\Delta + k - 1) \sum_{i=1}^n d_i x_i^2 + 2l \sum_{v_p v_q \notin E(G)} x_p x_q \\
 &\leq (2\Delta + k)q - \frac{\Delta + k - 1}{2}q + l(\Delta + n - 1 - q) \\
 &= \frac{1}{2}(3\Delta + k - 2l + 1)q + l(\Delta + n - 1).
 \end{aligned}$$

Solving the inequality gives the upper bound

$$q(G) \leq \frac{1}{4} \left[3\Delta + k - 2l + 1 + \sqrt{(3\Delta + k - 2l + 1)^2 + 16l(\Delta + n - 1)} \right].$$

If the upper bound of (1.1) is attained then all inequalities in the above argument must be equalities. In particular, from (3.2) and $x_i > 0$ for $1 \leq i \leq n$, we have that each pair of adjacent vertices in G has exactly k common neighbors and each pair of non-adjacent vertices in G has exactly l common neighbors. Moreover, by (3.3), G must be Δ -regular. Thus, G must be a strongly regular graph with parameters (Δ, k, l) . \square

Proof of Theorem 1.7. By Lemma 2.2, let w be a vertex of G such that

$$d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) = \max_{u \in V(G)} \left\{ d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v) \right\}.$$

Then

$$q(G) \leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i).$$

Note first that if $d(w) \leq s + t - 1$, then

$$\begin{aligned} q(G) &\leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) \leq d(w) + \Delta(G) \\ &\leq s + t - 1 + n - 1 = s + t + n - 2 \\ &\leq n + (s - t + 1)^{1/t} n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3. \end{aligned}$$

Therefore, we shall assume that $s + t - 1 \leq d(w) \leq n - 1$. Let U and W be disjoint sets satisfying $|U| = d(w)$ and $|W| = n - 1$, and let φ_U and φ_W be bijections

$$\varphi_U : U \rightarrow \Gamma(w), \varphi_W : W \rightarrow V(G) \setminus \{w\}.$$

Define a bipartite graph H with vertex classes U and W by joining $u \in U$ and $v \in W$ whenever $\{\varphi_U(u), \varphi_W(v)\} \in E(G)$.

Then we can get that H does not contain a copy of $K_{s-1,t}$ with $s - 1$ vertices in W and t vertices in U . Indeed, the map $\psi : V(H) \rightarrow V(G)$ defined as

$$\psi(x) = \begin{cases} \varphi_U(x), & \text{if } x \in U, \\ \varphi_W(x), & \text{if } x \in W. \end{cases}$$

is a homomorphism of H into $G - w$. Suppose to the contrary that $F \subset H$ is a copy of $K_{s-1,t}$ with a set of S of $s - 1$ vertices in W and a set of T of t vertices in U . Clearly S and T are the vertex classes of F . Note that $\psi(F)$ is a copy of $K_{s-1,t}$ in $G - w$, and $\psi(S) = \varphi_W(S) \subset V(G) \setminus \{w\}$ and $\psi(T) = \varphi_U(T) \subset \Gamma_G(w)$ are the vertex classes of $\psi(F)$ of size $s - 1$ and size t , respectively. Now, adding w to $\psi(F)$, we see that G contains a $K_{s,t}$, a contradiction proving the claim.

Suppose that $0 \leq k \leq \min\{s, t\} - 2$. Setting $k' = k - 1, s' = s - 1, t' = t, A = W, B = U$, then from Lemma 2.1, we have

$$\begin{aligned} e(H) &\leq (s - k - 1)^{1/t} |U| |W|^{1-1/t} + (k - 1) |U| + (t - 1) |W|^{1+(k-1)/t} \\ &= (s - k - 1)^{1/t} d(w) n^{1-1/t} + (k - 1) d(w) + (t - 1) (n - 1)^{1+(k-1)/t}. \end{aligned}$$

On the other hand, we have that

$$e(H) = \sum_{v \in \Gamma(w)} d(v) - d(w),$$

and so,

$$\sum_{v \in \Gamma(w)} d(v) \leq ((s - k - 1)^{1/t} n^{1-1/t} + k) d(w) + (t - 1) (n - 1)^{1+(k-1)/t}.$$

From Lemma 2.2, we have

$$\begin{aligned} q(G) &\leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) \\ &\leq d(w) + \frac{(t - 1) (n - 1)^{1+(k-1)/t}}{d(w)} + (s - k - 1)^{1/t} n^{1-1/t} + k. \end{aligned}$$

Since the function

$$f(x) = x + \frac{(t - 1) (n - 1)^{1+(k-1)/t}}{x}$$

is convex for $x > 0$, its maximum in any closed interval is attained at one of the endpoints of the interval. In the case $s + t - 1 \leq d(w) \leq n - 1$, then,

$$\begin{aligned} q(G) &\leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) \\ &\leq \max \left\{ s + t - 1 + \frac{(t-1)(n-1)^{1+(k-1)/t}}{s+t-1}, n-1 + \frac{(t-1)(n-1)^{1+(k-1)/t}}{n-1} \right\} \\ &\quad + (s-k-1)^{1/t} n^{1-1/t} + k \\ &\leq (s-k-1)^{1/t} n^{1-1/t} + k + \frac{(t-1)(n-1)^{1+(k-1)/t}}{n-1} + n-1 \\ &= (s-k-1)^{1/t} n^{1-1/t} + k + (t-1)(n-1)^{(k-1)/t} + n-1. \end{aligned}$$

Now, if $s \geq t \geq 3$, setting $k = t - 2$, we obtain

$$q(G) \leq n + (s-t+1)^{1/t} n^{1-1/t} + (t-1)(n-1)^{1-3/t} + t-3. \quad \square$$

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