Electronic Journal of Linear Algebra

Volume 30 *Volume 30 (2015)* Article 25

2015

Minimum ranks of sign patterns via sign vectors and duality

Marina Arav
Georgia State University, marav@gsu.edu

Frank Hall
Georgia State University, fhall@gsu.edu

Zhongshan Li
Georgia State University, zli@gsu.edu

Hein van der Holst
Georgia State University, hvanderholst@gsu.edu

John Henry Sinkovic
Georgia State University, johnsinkovic@gmail.com

See next page for additional authors

Authors
Marina Arav, Frank Hall, Zhongshan Li, Hein van der Holst, John Henry Sinkovic, and Lihua Zhang

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

ELA
http://math.technion.ac.il/iic/ela

MINIMUM RANKS OF SIGN PATTERNS VIA SIGN VECTORS AND DUALITY*

MARINA ARAV†, FRANK J. HALL†, ZHONGSHAN LI†, HEIN VAN DER HOLST†, JOHN SINKOVIC†, AND LIHUA ZHANG†

Abstract. A *sign pattern matrix* is a matrix whose entries are from the set $\{+, -, 0\}$. The minimum rank of a sign pattern matrix A is the minimum of the ranks of the real matrices whose entries have signs equal to the corresponding entries of A. It is shown in this paper that for any $m \times n$ sign pattern A with minimum rank $n-2$, rational realization of the minimum rank is possible. This is done using a new approach involving sign vectors and duality. It is shown that for each integer $n \geq 9$, there exists a nonnegative integer m such that there exists an $m \times n$ sign pattern matrix with minimum rank $n-3$ for which rational realization is not possible. A characterization of $m \times n$ sign patterns A with minimum rank $n-1$ is given (which solves an open problem in Brualdi et al. [R. Brualdi, S. Fallat, L. Hogben, B. Shader, and P. van den Driessche. Final report: Workshop on Theory and Applications of Matrices Described by Patterns. Banff International Research Station, Jan. 31 – Feb. 5, 2010.]), along with a more general description of sign patterns with minimum rank r, in terms of sign vectors of certain subspaces. Several related open problems are stated along the way.

Key words. Sign pattern matrix, Sign vectors, Minimum rank, Maximum rank, Rational realization, Oriented matroid duality.

AMS subject classifications. 15B35, 15B36, 52C40.

1. Introduction. An important part of combinatorial matrix theory is the study of sign pattern matrices, which has been the focus of extensive research for the last 50 years ([8], [15]). A sign pattern matrix is a matrix whose entries are from the set $\{+, -, 0\}$. For a real matrix B, $\operatorname{sign}(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-, 0$). For a sign pattern matrix A, the sign pattern class (also known as the qualitative class) of A, denoted $Q(A)$, is defined as

$$Q(A) = \{B : B \text{ is a real matrix and } \operatorname{sign}(B) = A\}.$$

The minimum rank of a sign pattern matrix A, denoted $\operatorname{mr}(A)$, is the minimum of the ranks of the real matrices in $Q(A)$. Determination of the minimum rank of a sign

*Received by the editors on December 26, 2013. Accepted for publication on June 27, 2015. Handling Editor: Bryan L. Shader.

†Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, USA (marav@gsu.edu).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

ELA
http://math.technion.ac.il/iic/ela

pattern matrix in general is a longstanding open problem (see [17]) in combinatorial matrix theory. Recently, there has been a number of papers concerning this topic, for example [1]–[12], [14]–[22]. In particular, matrices realizing the minimum rank of a sign pattern have applications in the study of neural networks [10] and communication complexity [22].

More specifically, in the study of communication complexity in computer science, $(+,-)$ sign pattern matrices arise naturally; the minimum rank (also known as the *sign-rank*) of the sign patterns plays an important role, as the minimum rank of a corresponding sign pattern matrix essentially determines the unbounded-error randomized communication complexity of a function.

In mathematics, rational realization is an important theme. For example, the study of the existence of rational (or integer) solutions of Diophantine equations is well-known. Steinitz's celebrated theorem stating that the combinatorial type of every 3-polytope is rationally realizable is a far-reaching result [27]. In combinatorics, rational realizability of certain point-line configurations is an important problem ([13, 25]); this is closely related to the realizability of the minimum rank of a sign pattern with minimum rank 3 [18]. More generally, the rational realizability of the minimum rank of a sign pattern matrix is equivalent to the rational realizability of a certain point-hyperplane configuration.

In [1], several classes of sign patterns A for which rational realization of the minimum rank is guaranteed are identified, such as when every entry of A is nonzero, or the minimum rank of A is at most 2, or the minimum rank of A is at least $n-1$, where A is $m \times n$. It has been shown in [19], through the use of a result in projective geometry, that rational realization of the minimum rank is not always possible. Specifically, in [19], the authors showed that there exists a 12×12 sign pattern matrix with minimum rank 3 but there is no rational realization of rank 3 within the qualitative class of the sign pattern. Independently, Berman et al. [5] also provided an example of a sign pattern for which the rational minimum rank is strictly greater than the minimum rank over the reals. Both of these papers use techniques based on matroids. More recently, Jing et al. [18] found a 9×9 sign pattern matrix with minimum rank 3 whose rational minimum rank is 4.

We note that Li et al. [20] showed that for every $n \times m$ sign pattern with minimum rank 2, there is an integer matrix in its sign pattern class each of whose entries has absolute value at most $2n-3$ that achieves the minimum rank.

One goal of this paper is to show that for any $m \times n$ sign pattern A with minimum rank $n-2$, rational realization of the minimum rank is possible. This is done using a new approach involving sign vectors and duality. Furthermore, it is shown that for each integer $n \geq 9$, there exists a nonnegative integer m such that there exists an

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

ELA
http://math.technion.ac.il/iic/ela

$n \times m$ sign pattern matrix with minimum rank $n-3$ for which rational realization is not possible.

Another goal is to use sign vectors of subspaces to investigate the minimum ranks of sign patterns further. In particular, a characterization of $m \times n$ sign patterns A with minimum rank $n-1$ is given, which solves an open problem posed in Brualdi et al. [6]. We also obtain a characterization of L-matrices using sign vectors and a more general description of sign patterns with minimum rank r, in terms of sign vectors of certain subspaces. Several related open problems are also discussed.

2. Sign vectors and duality. For any vector $x \in \mathbb{R}^n$, we define the *sign vector* of x, $\operatorname{sign}(x) \in \{+,-,0\}^n$, by

$$\operatorname{sign}(x)_i = \begin{cases} + & \text{if } x_i > 0, \\ 0 & \text{if } x_i = 0, \\ - & \text{if } x_i < 0. \end{cases}$$

For any subspace $L \subseteq \mathbb{R}^n$, we define the *set of sign vectors* of L as

$$\operatorname{sign}(L) = \{\operatorname{sign}(x) \mid x \in L\}.$$

Observation 2.1. If K and L are subspaces of $\mathbb{R}^n$ with $\operatorname{sign}(K) = \operatorname{sign}(L)$, then $\dim(K) = \dim(L)$.

Indeed, consider the sign vectors of the columns of the reduced column echelon form of a matrix whose columns form a basis of K. Such sign vectors are in $\operatorname{sign}(K)$ and hence also in $\operatorname{sign}(L)$. It follows that $\dim(L) \geq \dim(K)$. By reversing the roles of K and L, we get the reverse inequality.

For a subspace $L \subseteq \mathbb{R}^n$, as usual, $L^\perp = \{x \in \mathbb{R}^n \mid x^T y = 0 \text{ for all } y \in L\}$ denotes the *orthogonal complement* of L.

Two sign vectors $c, x \in \{+,-,0\}^n$ are said to be *orthogonal*, written as $c \perp x$, if one of the following two conditions holds:

1. for each i, we have $c_i = 0$ or $x_i = 0$, or
2. there are indices i, j with $c_i = x_i \neq 0$ and $c_j = -x_j \neq 0$.

For a set of sign vectors $S \subseteq \{+,-,0\}^n$, the *orthogonal complement* of S is

$$S^\perp = \{c \in \{+,-,0\}^n \mid c \perp x = 0 \text{ for all } x \in S\}.$$

Notice that if $c, x \in \mathbb{R}^n$ and $c^T x = 0$, then $\operatorname{sign}(c) \perp \operatorname{sign}(x) = 0$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

ELA
http://math.technion.ac.il/iic/ela

We will use the following theorem, a proof of which can be found in Ziegler [27].

THEOREM 2.2. [Duality of oriented matroids] *For any subspace* $L \subseteq \mathbb{R}^n$*,*

$$\operatorname{sign}(L)^{\perp} = \operatorname{sign}(L^{\perp}).$$

A subspace $L \subseteq \mathbb{R}^n$ is called *rational* if L has a basis consisting of rational vectors.

LEMMA 2.3. *Let L be a rational subspace of $\mathbb{R}^n$. For any sign vector $s \in \operatorname{sign}(L)$, there exists a rational vector $x \in L$ such that* $\operatorname{sign}(x) = s$.

Proof. Let $s \in \operatorname{sign}(L)$. Let $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix}$ be a matrix whose columns form a rational basis for L, let $N = \{i \ : \ s_i = 0\}$ and let $V = \{z \in \mathbb{R}^k \ : \ (Bz)_i = 0 \text{ for all } i \in N\}$. Since $\{b_1, b_2, \ldots, b_k\}$ is a rational basis for L, V has a rational basis $\{c_1, c_2, \ldots, c_t\}$. There exists a vector $y \in V$ such that $\operatorname{sign}(By) = s$. Let $a = \begin{bmatrix} a_1 & a_2 & \cdots & a_t \end{bmatrix}^T \in \mathbb{R}^t$ such that $a_1c_1 + a_2c_2 + \cdots + a_tc_t = y$. There exists a sequence of rational vectors $a(n) = \begin{bmatrix} a_1(n) & a_2(n) & \cdots & a_t(n) \end{bmatrix}^T$, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} a(n) = a$. Define $y(n) = a_1(n)c_1 + a_2(n)c_2 + \cdots + a_t(n)c_t$. Then $\lim_{n\to\infty} y(n) = y$ and each $y(n)$ is a rational vector. Since $\lim_{n\to\infty} By(n) = By$, there exists an m such that $\operatorname{sign}(By(m)) = \operatorname{sign}(By) = s$. By letting $x = By(m)$, the statement of the lemma is proved. $\square$

THEOREM 2.4. *For all nonnegative integers m and r, the following are equivalent:*

1. *Rational realization of the minimum rank is possible for every $m \times n$ sign pattern matrix with minimum rank r.*
2. *For every subspace $L \subseteq \mathbb{R}^m$ with dimension r, there exists a rational subspace $K \subseteq \mathbb{R}^m$ with* $\operatorname{sign}(K) = \operatorname{sign}(L)$.

Proof. Suppose that rational realization of the minimum rank is possible for every $m \times n$ sign pattern matrices with minimum rank r. Let $L \subseteq \mathbb{R}^m$ have dimension r. Let $s_1, \ldots, s_k$ be all sign vectors in $\operatorname{sign}(L)$. Let A be the $m \times k$ sign pattern matrix whose ith column is s_i. Then $\operatorname{mr}(A) \leq r$, as for each sign vector s_i we can choose a vector $x_i \in L$ with $\operatorname{sign}(x_i) = s_i$, and the matrix B, whose ith column is x_i, has rank r. Thus, there exists a rational matrix $C \in Q(A)$ with rank at most r. Let K be the column space of C. Then $\operatorname{sign}(K) = \operatorname{sign}(L)$ and K is rational.

Conversely, suppose that for each subspace $L \subseteq \mathbb{R}^m$ with dimension r there exists a rational subspace $K \subseteq \mathbb{R}^m$ with $\operatorname{sign}(K) = \operatorname{sign}(L)$. Let A be an $m \times n$ sign pattern matrix with $\operatorname{mr}(A) = r$ and let A_i denote the ith column of A. Since $\operatorname{mr}(A) = r$, there exists a matrix $F \in Q(A)$ with $\operatorname{rank}(F) = r$. Let $L \subseteq \mathbb{R}^m$ be the column space of F. Note that $\dim(L) = r$. By assumption, there exists a rational subspace K of $\mathbb{R}^m$ such that $\operatorname{sign}(L) = \operatorname{sign}(K)$. By Observation 2.1, $\dim(K) = r$. Since

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

ELA
http://math.technion.ac.il/iic/ela

$\text{sign}(L) = \text{sign}(K)$, there exist rational vectors $x_i \in K$, $i = 1, 2, \ldots, n$ such that $\text{sign}(x_i) = A_i$, by Lemma 2.3. Let $C = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$. Then $C \in Q(A)$, and C is a rational matrix. Since $\dim(K) = r$ and $\text{mr}(A) = r$, $\text{rank}\, C = r$. ☐

The next theorem can be found in [1] and [20].

Theorem 2.5. *Rational realization of the minimum rank is possible for every $m \times n$ sign pattern matrix A with minimum rank* 2.

The following theorem follows from the two preceding theorems.

Theorem 2.6. *For any subspace $L \subseteq \mathbb{R}^n$ with $\dim(L) = 2$, there exists a rational subspace $K \subseteq \mathbb{R}^n$ with $\dim(K) = 2$ and $\text{sign}(K) = \text{sign}(L)$.*

Since the solution space of a system of homogeneous linear equations with rational coefficients has a rational basis, it is clear that if $L \subseteq \mathbb{R}^n$ is a rational subspace, then $L^\perp$ is also a rational subspace.

Lemma 2.7. *For any subspace $L \subseteq \mathbb{R}^n$ with $\dim L = n-2$, there exists a rational subspace $K \subseteq \mathbb{R}^n$ with $\dim K = n-2$ such that $\text{sign}(K) = \text{sign}(L)$.*

Proof. By Theorem 2.6, there exists a rational subspace $M \subseteq \mathbb{R}^n$ with $\dim M = 2$ such that $\text{sign}(L^\perp) = \text{sign}(M)$. Let $K = M^\perp$. Then K is a rational subspace of $\mathbb{R}^n$ with $\dim K = n - 2$ and, by Theorem 2.2, $\text{sign}(K) = \text{sign}(M^\perp) = \text{sign}(M)^\perp = \text{sign}(L^\perp)^\perp = \text{sign}(L)$. ☐

By Theorem 2.4, and by considering the transpose if needed, we get the following result.

Theorem 2.8. *Rational realization of the minimum rank is possible for every $m \times n$ sign pattern matrix with minimum rank $m-2$ or $n-2$.*

From the 9×9 example given in [18], the next theorem follows immediately.

Theorem 2.9. *For each integer $n \geq 9$, there exists an $m \times n$ sign pattern matrix A with $\text{mr}(A) = 3$ for which no rational realization is possible.*

From Theorem 2.4 we then obtain the following corollary.

Corollary 2.10. *Let $n \geq 9$ be an integer. Then there exists a subspace $L \subseteq \mathbb{R}^n$ with $\dim L = 3$ such that there is no rational subspace $K \subseteq \mathbb{R}^n$ with $\dim K = 3$ and $\text{sign}(L) = \text{sign}(K)$.*

The following result then follows from Theorem 2.2.

Lemma 2.11. *Let $n \geq 9$ be an integer. There exists a subspace $M \subseteq \mathbb{R}^n$ with $\dim M = n - 3$ such that there is no rational subspace $K \subseteq \mathbb{R}^n$ with $\text{sign}(K) =$*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

ELA
http://math.technion.ac.il/iic/ela

$\operatorname{sign}(M)$.

Proof. By Corollary 2.10, there exists a subspace $L \subseteq \mathbb{R}^n$ with $\dim L = 3$ such that there is no rational subspace $K \subseteq \mathbb{R}^n$ with $\dim K = 3$ and $\operatorname{sign}(L) = \operatorname{sign}(K)$. Let $M = L^\perp$. Suppose for a contradiction that there exists a rational subspace $S \subseteq \mathbb{R}^n$ with $\dim(S) = n-3$ and $\operatorname{sign}(S) = \operatorname{sign}(M)$. Then $S^\perp$ is a rational subspace of $\mathbb{R}^n$ with $\dim(S^\perp) = 3$. By Theorem 2.2, $\operatorname{sign}(S^\perp) = \operatorname{sign}(S)^\perp = \operatorname{sign}(M)^\perp = \operatorname{sign}(M^\perp) = \operatorname{sign}(L)$. This contradicts the assumption that there is no rational subspace $K \subseteq \mathbb{R}^n$ with $\operatorname{sign}(L) = \operatorname{sign}(K)$. ◻

Another application of Theorem 2.4 gives the following fact.

THEOREM 2.12. *For each integer $n \geq 9$, there exists a nonnegative integer m such that there exists an $m \times n$ sign pattern matrix with minimum rank $n-3$ for which rational realization is not possible.*

However, this leaves open the following question.

PROBLEM 2.13. Is it true that for each integer $n \geq 9$, there exists an $n \times n$ sign pattern matrix with minimum rank $n-3$ for which rational realization is not possible?

Another natural question is the following.

PROBLEM 2.14. Is it true that for every 3-dimensional subspace L of $\mathbb{R}^8$, there is a rational subspace K such that $\operatorname{sign}(L) = \operatorname{sign}(K)$?

There are connections between rational realization of the minimum ranks of sign patterns and the existence of rational solutions of certain matrix equations, as indicated in [3]. Using this connection and Theorem 2.8 for the case of $\operatorname{mr}(A) = n-2$, we are able to prove the following result.

THEOREM 2.15. *Suppose that B, C and E are real matrices such that $BC = E$. If E has either 2 rows or 2 columns, then there exist rational matrices $\tilde{B}$, $\tilde{C}$ and $\tilde{E}$ such that $\operatorname{sign}(\tilde{B}) = \operatorname{sign}(B)$, $\operatorname{sign}(\tilde{C}) = \operatorname{sign}(C)$, $\operatorname{sign}(\tilde{E}) = \operatorname{sign}(E)$, and $\tilde{B}\tilde{C} = \tilde{E}$.*

Proof. Without loss of generality, assume that E has two columns. Consider the 2×2 block matrix

$$M = \begin{bmatrix} I_n & C \\ B & E \end{bmatrix}.$$

Observe that M has $n+2$ columns and the Schur complement of I_n in M is 0. Hence, $\operatorname{rank}(M) = n$. It follows that the minimum rank of the sign pattern $\operatorname{sign}(M)$ is n. Hence, from Theorem 2.8, there is a rational matrix

$$\tilde{M} = \begin{bmatrix} D & C_1 \\ B_1 & E_1 \end{bmatrix}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

ELA
http://math.technion.ac.il/iic/ela

of rank n in $Q(\mathrm{sign}(M))$.

It follows that the Schur complement of D in $\tilde{M}$ is $E_1 - B_1D^{-1}C_1 = 0$. The rational matrices $\tilde{B} = B_1$, $\tilde{C} = D^{-1}C_1$ and $\tilde{E} = E_1$ clearly satisfy the desired properties. □

Since the zero entries of the matrix E in a real matrix equation $BC = E$ are the major obstructions for the existence of rational solutions within the same corresponding sign pattern classes, the preceding theorem suggests the following conjecture.

Conjecture 2.16. Let B, C and E be real matrices such that $BC = E$. If all the zero entries of E are contained in a submatrix with either 2 rows or 2 columns, then there exist rational matrices $\tilde{B}$, $\tilde{C}$, and $\tilde{E}$, such that $\mathrm{sign}(\tilde{B}) = \mathrm{sign}(B)$, $\mathrm{sign}(\tilde{C}) = \mathrm{sign}(C)$, $\mathrm{sign}(\tilde{E}) = \mathrm{sign}(E)$, and $\tilde{B}\tilde{C} = \tilde{E}$.

3. Further results on minimum ranks and sign vectors of subspaces. We now investigate the minimum ranks of sign patterns further using sign vectors and duality.

Similar to the concept of $\mathrm{mr}(A)$, the *maximum rank* of a sign pattern matrix A, denoted $\mathrm{MR}(A)$, is the maximum of the ranks of the real matrices in $Q(A)$. It is well-known that $\mathrm{MR}(A)$ is the maximum number of nonzero entries of A with no two of the nonzero entries in the same row or in the same column. By a theorem of König, the minimal number of lines (namely, rows and columns) in A that cover all of the nonzero entries of A is equal to the maximal number of nonzero entries in A, no two of which are on the same line. This common number is also called the *term rank* [7, 15]. In contrast to the rational realization problem of the minimum rank, we note that through diagonal dominance, it can be easily seen that for every sign pattern matrix A, $\mathrm{MR}(A)$ can always be achieved by a rational matrix.

Sign pattern matrices A that require a unique rank (namely, $\mathrm{MR}(A) = \mathrm{mr}(A)$) were characterized by Hershkowitz and Schneider in [16].

It is shown in [2] that rational realization of the minimum rank of A is always possible if $\mathrm{MR}(A) - \mathrm{mr}(A) = 1$.

In [2], the study of sign patterns A with $\mathrm{MR}(A) - \mathrm{mr}(A) = 1$ is reduced to the study of $m \times n$ sign patterns A such that $\mathrm{MR}(A) = n$ and $\mathrm{mr}(A) = n - 1$. In this connection, using sign vectors and duality, we obtain the following characterization of the $m \times n$ sign patterns A such that $\mathrm{MR}(A) = n$ and $\mathrm{mr}(A) = n - 1$ (which was raised as an open problem in [6]). Since the condition $\mathrm{MR}(A) = n$ is easily checked, it suffices to characterize $m \times n$ sign patterns A with $\mathrm{mr}(A) = n - 1$.

For an $m \times n$ sign pattern or real matrix, we denote by $\mathrm{row}(A)$ the set of the row

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

ELA
http://math.technion.ac.il/iic/ela

vectors of A.

THEOREM 3.1. *Let A be an $m \times n$ sign pattern. Then* $\mathrm{mr}(A) = n - 1$ *if and only if the following two conditions hold:*

(a) There is a nonzero sign vector $x \in \{+, -, 0\}^n$ such that $\mathrm{row}(A) \subseteq x^\perp$.
(b) For every two-dimensional subspace L of $\mathbb{R}^n$, $\mathrm{row}(A) \not\subseteq \mathrm{sign}(L)^\perp$.

Proof. Assume that $\mathrm{mr}(A) = n - 1$. Then there is a real matrix $A_1 \in Q(A)$ such that $\mathrm{rank}(A_1) = n - 1$. Let v_0 be a nonzero vector in the null space $\mathrm{Null}(A_1)$ and let $x = \mathrm{sign}(v_0)$. Then it is clear that $\mathrm{row}(A) \subseteq x^\perp$, namely, (a) holds. Further, consider any two-dimensional subspace L of $\mathbb{R}^n$. Suppose that $\mathrm{row}(A) \subseteq \mathrm{sign}(L)^\perp$. Then by Theorem 2.2, $\mathrm{row}(A) \subseteq \mathrm{sign}(L^\perp)$. Since $L^\perp$ is a subspace of dimension $n - 2$ and every row vector of A is contained in $\mathrm{sign}(L^\perp)$, it is clear that the span of the real vectors in $L^\perp$ whose sign vectors agree with the rows of A is a subspace of dimension at most $n - 2$, that is to say, there is a real matrix $A_2 \in Q(A)$ of rank at most $n - 2$, which contradicts the assumption that $\mathrm{mr}(A) = n - 1$.

Conversely, assume that (a) and (b) hold. Let x_1 be the $(1, -1, 0)$ vector in $Q(x)$. From (a), it is easily seen that there is a real matrix $A_1 \in Q(A)$ such that $A_1 x_1 = 0$. It follows that $\mathrm{rank}(A_1) \le n - 1$ and hence $\mathrm{mr}(A) \le n - 1$. Suppose that $\mathrm{mr}(A) \le n - 2$. Then there is a matrix $A_2 \in Q(A)$ with $\mathrm{rank}(A_2) \le n - 2$. It follows that $\mathrm{Null}(A_2)$ has dimension at least 2. Hence, $\mathrm{Null}(A_2)$ contains a subspace $L \in \mathbb{R}^n$ of dimension 2. Since $L \subseteq \mathrm{Null}(A_2)$, we have that $\mathrm{row}(A_2) \subseteq L^\perp$, and hence $\mathrm{row}(A) = \mathrm{sign}(\mathrm{row}(A_2)) \subseteq \mathrm{sign}(L^\perp)$. By Theorem 2.2, $\mathrm{sign}(L^\perp) = \mathrm{sign}(L)^\perp$, and so $\mathrm{row}(A) \subseteq \mathrm{sign}(L)^\perp$, contradicting (b). Thus, $\mathrm{mr}(A) = n - 1$. $\square$

We note that for an $m \times n$ sign pattern matrix A, the condition that each row vector of A is in $\{u, v\}^\perp$ for some two nonzero sign vectors $u, v \in \{+, -, 0\}^n$ with $u \ne \pm v$ does not imply that $\mathrm{mr}(A) \le n - 2$, as the following example shows.

EXAMPLE 3.2. Let $A = \begin{bmatrix} + & + & + \\ 0 & + & + \end{bmatrix}$, $u = \begin{bmatrix} + \\ + \\ - \end{bmatrix}$, and $v = \begin{bmatrix} 0 \\ + \\ - \end{bmatrix}$. It is obvious that each row vector of A is in $\{u, v\}^\perp$ for the two nonzero sign vectors $u, v \in \{+, -, 0\}^3$ with $u \ne \pm v$. But clearly, $\mathrm{mr}(A) = 2$, so $\mathrm{mr}(A) \not\le n - 2 = 1$.

Using a similar argument as in the proof of Theorem 3.1, we can show the following more general result that characterizes sign patterns with minimum rank r, for each possible $r \ge 1$.

THEOREM 3.3. *Let A be an $m \times n$ sign pattern and let* $\mathrm{row}(A)$ *denote the set of the row vectors of A, and let r be a any integer such that $1 \le r \le \min\{m, n\}$. Then* $\mathrm{mr}(A) = r$ *if and only if the following two conditions hold:*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

(a) There is a subspace $L \subseteq \mathbb{R}^n$ *with* $\dim(L) = r$ *such that* $\mathrm{row}(A) \subseteq \mathrm{sign}(L)$.
(b) For every subspace V *of* $\mathbb{R}^n$ *with* $\dim(V) = r-1$, $\mathrm{row}(A) \not\subseteq \mathrm{sign}(V)$.

In particular, the above theorem gives a characterization of L-matrices (namely, $m \times n$ sign patterns A with $\mathrm{mr}(A) = n$). Note that every subset of $\{+,-,0\}^n$ is of course contained in $\mathrm{sign}(\mathbb{R}^n)$ and Theorem 2.2 gives us that for every subspace V of $\mathbb{R}^n$ of dimension $n-1$, $\mathrm{sign}(V) = \mathrm{sign}(x)^\perp$ for each nonzero vector x in $V^\perp$.

Corollary 3.4. *Let A be an $m \times n$ sign pattern and let* $\mathrm{row}(A)$ *denote the set of the rows of A. Then,* $\mathrm{mr}(A) = n$ *if and only if for every nonzero sign vector* $x \in \{+,-,0\}^n$, $\mathrm{row}(A) \not\subseteq x^\perp$.

The last characterization of L-matrices can be seen to be equivalent to the characterization of such matrices found in [8] (where a sign pattern is defined to be an L-matrix if the minimum rank is equal to its number of rows).

Let $S_{k,n}$ (respectively, $s_{k,n}$) denote the maximum cardinality (respectively, minimum cardinality) of $\mathrm{sign}(L)$ as L runs over all k-dimensional subspaces of $\mathbb{R}^n$. In other words,

$$S_{k,n} = \max_{\substack{L\subseteq\mathbb{R}^n\\ \dim(L)=k}} |\,\mathrm{sign}(L)|, \quad s_{k,n} = \min_{\substack{L\subseteq\mathbb{R}^n\\ \dim(L)=k}} |\,\mathrm{sign}(L)|.$$

For every subspace L, as the nonzero vectors in $\mathrm{sign}(L)$ occur in disjoint pairs of vectors that are negatives of each other, it is clear that $|\,\mathrm{sign}(L)|$ is odd. Thus, $S_{k,n}$ and $s_{k,n}$ are always odd. For each k $(0 \le k \le n-1)$, since every k-dimensional subspace of $\mathbb{R}^n$ is contained in a subspace of dimension $k+1$, it is clear that $S_{k,n} \le S_{k+1,n}$ and $s_{k,n} \le s_{k+1,n}$.

Obviously, $S_{0,n} = s_{0,n} = 1$, $S_{1,n} = s_{1,n} = 3$, and $S_{n,n} = s_{n,n} = 3^n$. By considering the reduced row echelon form of a matrix A whose rows form a basis for a k-dimensional subspace L of $\mathbb{R}^n$ and observing that the components of the vectors in L in the pivot columns of A are independent and arbitrary, it can be seen that $s_{k,n} \ge 3^k$, for each $k,\ 1 \le k \le n$. For the k-dimensional subspace L spanned by the standard vectors $e_1, e_2, \ldots, e_k$, it can be seen that equality in the last inequality can be achieved. Thus, $s_{k,n} = 3^k$, for each $k,\ 1 \le k \le n$. We record this result below.

Theorem 3.5. *Let $n \ge 2$. Then $s_{k,n} = 3^k$, for each $k,\ 1 \le k \le n$.*

Let L be a k-dimensional linear subspace of $\mathbb{R}^n$. Let B be a matrix whose columns form a basis of L. For each $1 \le j \le n$, let H_j be the orthogonal complement of the jth row of B in $\mathbb{R}^k$. The central hyperplane arrangement $\{H_1, \ldots, H_n\}$ in $\mathbb{R}^k$ partitions $\mathbb{R}^k$ into disjoint, relatively open cells of dimensions 0 through k, with each cell corresponding to precisely one sign vector in $\mathrm{sign}(L)$, see [21, 24, 26, 27]. It can be seen that $S_{k,n}$ is equal to the total number of cells of a generic central hyperplane

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

ELA
http://math.technion.ac.il/iic/ela

arrangement $\{H_1, \ldots, H_n\}$ in $\mathbb{R}^k$. For $n \geq k$, adding a hyperplane H_{n+1} to a generic central hyperplane arrangement $\{H_1, \ldots, H_n\}$ in generic position increases the total number of cells by $2(S_{k-1,n} - 1)$. This can been seen as follows. For $m > 1$, each relatively open cell of dimension m of the central hyperplane arrangement $\{H_1, \ldots, H_n\}$ that is intersected by H_{n+1} yields three relatively open cells in the hyperplane arrangement $\{H_1, \ldots, H_{n+1}\}$, one of dimension $m-1$, which is a relatively open cell of $\{H_1 \cap H_{n+1}, \ldots, H_n \cap H_{n+1}\}$, and two of dimension m. For $m \leq 1$, each relatively open cell of dimension m of the central hyperplane arrangement $\{H_1, \ldots, H_n\}$ is not cut by H_{n+1} and is a relatively open cell in the hyperplane arrangement $\{H_1, \ldots, H_{n+1}\}$. (Here, we used that $n \geq k$, so that there exists a cell of dimension 0 of the arrangement $\{H_1, \ldots, H_n\}$.) Since the number of relatively open cells of the arrangement $\{H_1 \cap H_{n+1}, \ldots, H_n \cap H_{n+1}\}$ in H_{n+1} is $S_{k-1,n}$, the increase of the total number of cells is $2(S_{k-1,n} - 1)$. Therefore the following recursion formula holds for each $n \geq 2$ and each $1 \leq k < n$:

$$S_{k,n} = S_{k,n-1} + 2(S_{k-1,n-1} - 1).$$

As pointed out by Richard Stanley, the total number of cells of a generic central hyperplane arrangement $\{H_1, \ldots, H_n\}$ of $\mathbb{R}^k$ can be computed using the intersection lattice approach described in [24, 26]. The resulting formula is given below.

Theorem 3.6. *For each $n \geq 2$ and each $1 \leq k \leq n$,*

$$S_{k,n} = 1 + \sum_{i=0}^{k-1} 2^i \binom{n}{i} + \sum_{i=1}^{k} \binom{n}{k-i} \binom{n-k+i-1}{i-1}.$$

Example 3.7.

1. $S_{1,n} = 3$.
2. $S_{2,n} = 4n + 1$.
3. $S_{3,n} = 4n^2 - 4n + 3$.
4. $S_{4,n} = \frac{8}{3}n^3 - 8n^2 + \frac{28}{3}n + 1$.
5. $S_{5,n} = \frac{4}{3}n^4 - 8n^3 + \frac{56}{3}n^2 - 12n + 3$.
6. $S_{6,n} = \frac{8}{15}n^5 - \frac{16}{3}n^4 + \frac{64}{3}n^3 - \frac{104}{3}n^2 + \frac{332}{15}n + 1$.

Consequently, we have the following results on sign pattern matrices.

Theorem 3.8. *Let A be an $m \times n$ sign pattern matrix.*

1. *If $mr(A) = 1$, then $|row(A)| \leq 3$.*
2. *If $mr(A) = 2$, then $|row(A)| \leq 4n + 1$.*
3. *If $mr(A) = 3$, then $|row(A)| \leq 4n^2 - 4n + 3$.*
4. *If $mr(A) = 4$, then $|row(A)| \leq \frac{8}{3}n^3 - 8n^2 + \frac{28}{3}n + 1$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

ELA
http://math.technion.ac.il/iic/ela

5. *If* $mr(A) = 5$, *then* $|row(A)| \leq \frac{4}{3}n^4 - 8n^3 + \frac{56}{3}n^2 - 12n + 3$.
6. *If* $mr(A) = 6$, *then* $|row(A)| \leq \frac{8}{15}n^5 - \frac{16}{3}n^4 + \frac{64}{3}n^3 - \frac{104}{3}n^2 + \frac{332}{15}n + 1$.

Final remarks. The main results of this paper were contained in a preliminary version of the paper, and were presented at the 2013 ILAS conferences by one of the coauthors. In 2014, we noticed the related paper by Shitov [23] that has some overlap. Unfortunately, the proof of a key lemma (Lemma 3.5) in Shitov's paper has a logical gap.

Acknowledgment. We thank the referee for a thorough reading and for the many valuable suggestions and comments. We thank Richard Stanley for providing us with the explicit formula for $S_{k,n}$ in Theorem 3.6.

REFERENCES

[1] M. Arav, F.J. Hall, S. Koyuncu, Z. Li, and B. Rao. Rational realizations of the minimum rank of a sign pattern matrix. *Linear Algebra Appl.*, 409:111–125, 2005.
[2] M. Arav, F.J. Hall, Z. Li, A. Merid, and Y. Gao. Sign patterns that require almost unique rank. *Linear Algebra Appl.*, 430:7–16, 2009.
[3] M. Arav, F. Hall, Z. Li, and B. Rao. Rational solutions of certain matrix equations. *Linear Algebra Appl.*, 430:660–663, 2009.
[4] F. Barioli, S.M. Fallat, H.T. Hall, D. Hershkowitz, L. Hogben, H. van der Holst, and B. Shader. On the minimum rank of not necessarily symmetric matrices: a preliminary study. *Electron. J. Linear Algebra*, 18:126–145, 2009.
[5] A. Berman, S. Friedland, L. Hogben, U.G. Rothblum, and B. Shader. Minimum rank of matrices described by a graph or pattern over the rational, real and complex numbers. *The Electron. J. Combin.*, 15: Research Paper 25, 2008.
[6] R. Brualdi, S. Fallat, L. Hogben, B. Shader, and P. van den Driessche. Final report: Workshop on Theory and Applications of Matrices Described by Patterns. Banff International Research Station, Jan. 31 – Feb. 5, 2010.
[7] R. Brualdi and H.J. Ryser. Combinatorial matrix theory. *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1991.
[8] R.A. Brualdi and B.L. Shader. *Matrices of Sign-Solvable Linear Systems.* Cambridge University Press, Cambridge, 1995.
[9] G. Chen, F.J. Hall, Z. Li, and B. Wei. On ranks of matrices associated with trees. *Graphs Combin.*, 19:323–334, 2003.
[10] P. Delsarte and Y. Kamp. Low rank matrices with a given sign pattern. *SIAM J. Discrete Math.*, 2:51–63, 1989.
[11] J. Forster. A linear lower bound on the unbounded error probabilistic communication complexity. *J. Comput. Syst. Sci.*, 65(4):612–625, 2002.
[12] R.L. Graham, M. Grötschel, and L. Lovász (Editors). *Handbook of Combinatorics.* MIT Press, 1996.
[13] B. Grünbaum. *Configurations of Points and Lines.* Graduate Studies in Mathematics, Vol. 103, Amer. Math. Soc., Providence, RI, 2009.
[14] F.J. Hall, Z. Li, and B. Rao. From Boolean to sign pattern matrices. *Linear Algebra Appl.*, 393:232–251, 2004.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 360-371, July 2015

ELA
http://math.technion.ac.il/iic/ela

[15] F.J. Hall and Z. Li. Sign pattern matrices. In: L. Hogben et al. (editors), *Handbook of Linear Algebra*, Chapman and Hall/CRC Press, Boca Raton, 2013.

[16] D. Hershkowitz and H. Schneider. Ranks of zero patterns and sign patterns. *Linear Multilinear Algebra*, 34:3–19, 1993.

[17] C.R. Johnson. Some outstanding problems in the theory of matrices. *Linear Multilinear Algebra*, 12:99–108, 1982.

[18] G. Jing, W. Gao, Y. Gao, F. Gong, Z. Li, Y. Shao, and L. Zhang. Sign patterns with minimum rank 3 and point-line configurations on the plane. Preprint, available at `http://arxiv.org/abs/1312.6162`.

[19] S. Kopparty and K.P.S. Bhaskara Rao. The minimum rank problem: A counterexample. *Linear Algebra Appl.*, 428:1761–1765, 2008.

[20] Z. Li, Y. Gao, M. Arav, F. Gong, W. Gao, F.J. Hall, and H. van der Holst. Sign patterns with minimum rank 2 and upper bounds on minimum ranks. *Linear Multilinear Algebra*, 61:895–908, 2013.

[21] J. Matoušek. *Lectures on Discrete Geometry.* Springer-Verlag, Berlin, 2002.

[22] A.A. Razborov and A.A. Sherstov. The sign rank of AC^0. *SIAM J. Comput.*, 39:1833–1855, 2010.

[23] Y. Shitov. Sign patterns of rational matrices with large rank. *European J. Comb.*, 42:107–111, 2014.

[24] R.P. Stanley. Hyperplane arrangements. In: E. Miller, V. Reiner, and B. Sturmfels (editors), *Geometric Combinatorics*, IAS/Park City Mathematics Series, Vol. 13, Amer. Math. Soc., Providence, RI, 389–496, 2007.

[25] B. Sturmfels and N. White. All 11_3 and 12_3-configurations are rational. *Aequationes Math.*, 39:254–260, 1990.

[26] T. Zaslavsky. Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes. *Mem. Amer. Math. Soc.*, 154, 1975.

[27] G.M. Ziegler. *Lectures on Polytopes.* Springer-Verlag, New York, 1994.