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Realizing Suleimanova-type Spectra via Permutative Matrices

Pietro Paparella

University of Washington - Bothell Campus, pietrop@uw.edu

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REALIZING SULEIMANOVA SPECTRA VIA
PERMUTATIVE MATRICES

PIETRO PAPARELLA†

Abstract. A permutative matrix is a square matrix such that every row is a permutation of the first row. A constructive version of a result attributed to Suleimanova is given via permutative matrices. A well-known result is strengthened by showing that all realizable spectra containing at most four elements can be realized by a permutative matrix or by a direct sum of permutative matrices. The paper concludes by posing a problem.

Key words. Suleimanova spectrum, Permutative matrix, Real nonnegative inverse eigenvalue problem.

AMS subject classifications. 15A18, 15A29, 15B99.

1. Introduction. Introduced by Suleimanova in [13], the longstanding real nonnegative inverse eigenvalue problem (RNIEP) is to determine necessary and sufficient conditions on a set \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R} \) so that \( \sigma \) is the spectrum an \( n \)-by-\( n \) entrywise nonnegative matrix.

If \( A \) is an \( n \)-by-\( n \) nonnegative matrix with spectrum \( \sigma \), then \( \sigma \) said to be realizable and the matrix \( A \) is called a realizing matrix for \( \sigma \). It is well-known that if \( \sigma \) is realizable, then

\[
s_k(\sigma) := \sum_{i=1}^{n} \lambda_i^k \geq 0, \quad \forall \, k \in \mathbb{N}
\]

\[
\rho(\sigma) := \max_{1 \leq i \leq n} |\lambda_i| \in \sigma.
\]

For additional background and results, see, e.g., [2, 9] and references therein.

A set \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R} \) is called a Suleimanova spectrum if \( s_1(\sigma) \geq 0 \) and \( \sigma \) contains exactly one positive element. Suleimanova [13] announced (and loosely proved) that every such spectrum is realizable. Fiedler [3] showed that every Suleimanova spectrum is symmetrically realizable (i.e., realizable by a symmetric nonnegative matrix), however, his proof is by induction and does not explicitly yield a realizing

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† Division of Engineering and Mathematics, University of Washington Bothell, Bothell, WA 98011-8246, USA (pietro@uw.edu).
matrix for all orders. In [6], Johnson and Paparella provide a constructive version of Fiedler’s result for Hadamard orders.

Friedland [4] and Perfect [10] proved Suleimanova’s result via companion matrices (for other proofs, see references in [4]). In particular, the coefficients $c_0, c_1, \ldots, c_{n-1}$ of the polynomial $p(t) := \prod_{k=1}^{n}(t - \lambda_k) = t^n + \sum_{k=0}^{n-1} c_k t^k$ are nonpositive so that the companion matrix of $p$ is nonnegative. As noted in [11, p. 1380], the construction of the companion matrix of $p$ requires evaluating the elementary symmetric functions at $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a computation with $O(2^n)$ complexity.

The computation of a realizing matrix for a realizable spectrum is of obvious interest for numerical purposes, but for many known theoretical results, a realizing matrix is not readily available. Indeed, according to Chu:

Very few of these theoretical results are ready for implementation to actually compute [the realizing] matrix. The most constructive result we have seen is the sufficient condition studied by Soules [12]. But the condition there is still limited because the construction depends on the specification of the Perron vector – in particular, the components of the Perron eigenvector need to satisfy certain inequalities in order for the construction to work. [1, p. 18].

In this work, we provide a constructive version of Suleimanova’s result via permutative matrices. The paper is organized as follows: Section 2 contains notation and definitions; Section 3 contains the main results; in Section 4, we show that if $\sigma = \{\lambda_1, \ldots, \lambda_n\}$, $n \leq 4$, satisfies (1.1) and (1.2), then $\sigma$ is realizable by a permutative matrix or by a direct sum of permutative matrices; and we conclude by posing a problem in Section 5.

2. Notation. The set of $m$-by-$n$ matrices with entries from a field $\mathbb{F}$ (in this paper, $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$) is denoted by $M_{m,n}(\mathbb{F})$ (when $m = n$, $M_{n,n}(\mathbb{F})$ is abbreviated to $M_n(\mathbb{F})$). For $A = [a_{ij}] \in M_n(\mathbb{C})$, $\sigma(A)$ denotes the spectrum of $A$.

The set of $n$-by-1 column vectors is identified with the set of all $n$-tuples with entries in $\mathbb{F}$ and thus denoted by $\mathbb{F}^n$. Given $x \in \mathbb{F}^n$, $x_i$ denotes the $i^{th}$ entry of $x$.

For the following, the size of each object will be clear from the context in which it appears:

- $I$ denotes the identity matrix;
- $e$ denotes the all-ones vector; and
- $J$ denotes the all-ones matrix, i.e., $J = ee^T$. 

DEFINITION 2.1. For $x \in \mathbb{C}^n$ and permutation matrices $P_1, \ldots, P_n \in M_n(\mathbb{R})$, a permutative matrix\(^1\) is any matrix of the form

$$
\begin{bmatrix}
 x^T \\
 (P_2 x)^T \\
 \vdots \\
 (P_n x)^T
\end{bmatrix} \in M_n(\mathbb{C}).
$$

According to Definition 2.1, all one-by-one matrices are considered permutative.

3. Main results. We begin with the following lemmas.

LEMMA 3.1. For $x \in \mathbb{C}^n$, let

$$
P = P_x :=
\begin{bmatrix}
 1 & 2 & \cdots & i & \cdots & n \\
 1 & x_1 & x_2 & \cdots & x_i & \cdots & x_n \\
 2 & x_2 & x_1 & \cdots & x_i & \cdots & x_n \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
 i & x_i & x_2 & \cdots & x_1 & \cdots & x_n \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
 n & x_n & x_2 & \cdots & x_i & \cdots & x_1
\end{bmatrix}
$$

where $P_{\alpha_i}$ is the permutation matrix corresponding to the permutation $\alpha_i$ defined by $\alpha_i(x) = (1 \, i), i = 2, \ldots, n$. Then $\sigma(P) = \{s, \delta_2, \ldots, \delta_n\}$, where $s := \sum_{i=1}^{n} x_i$ and $\delta_i := x_1 - x_i$, $i = 2, \ldots, n$.

Proof. Since every row sum of $P$ is $s$, it follows that $Pe = sc$, i.e., $s \in \sigma(P)$.

Since

$$
P - \delta_i I =
\begin{bmatrix}
 1 & 2 & \cdots & i & \cdots & n \\
 1 & x_i & x_2 & \cdots & x_i & \cdots & x_n \\
 2 & x_2 & x_i & \cdots & x_i & \cdots & x_n \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
 i & x_i & x_2 & \cdots & x_i & \cdots & x_n \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
 n & x_n & x_2 & \cdots & x_i & \cdots & x_1
\end{bmatrix},
$$

it follows that the homogeneous linear system $(P - \delta_i I)x = 0$ has a nontrivial solution (notice that the first and $i$th rows of $P - \delta_i I$ are identical). Thus, $\delta_i \in \sigma(P)$.

\(^1\)Terminology due to Charles R. Johnson.
Moreover, if
\[
v_i := \begin{bmatrix}
1 \\
\vdots \\
i-1 \\
i \\
i+1 \\
\vdots \\
n
\end{bmatrix}
\begin{bmatrix}
x_i \\
x_1 - s \\
x_i \\
x_i \\
x_i \\
x_i \\
x_i \\
x_i \\
x_i \\
\end{bmatrix}, \quad i = 2, \ldots, n
\]
then
\[
P v_i =
\begin{bmatrix}
1 \\
\vdots \\
i-1 \\
i \\
i+1 \\
\vdots \\
n
\end{bmatrix}
\begin{bmatrix}
x_i (s - x_i) + x_i (x_1 - s) \\
x_i (s - x_1) + x_1 (x_i - s) \\
x_i (s - x_i) + x_i (x_1 - s) \\
x_i (s - x_1) + x_1 (x_i - s) \\
x_i (s - x_i) + x_i (x_1 - s) \\
x_i (s - x_1) + x_1 (x_i - s) \\
x_i (s - x_i) + x_i (x_1 - s) \\
x_i (s - x_1) + x_1 (x_i - s) \\
x_i (s - x_i) + x_i (x_1 - s) \\
x_i (s - x_1) + x_1 (x_i - s)
\end{bmatrix}
= (x_1 - x_i) \begin{bmatrix}
x_1 - s \\
x_i \\
x_i \\
x_i \\
x_i \\
x_i \\
x_i \\
x_i \\
x_i \\
x_i \\
x_i 
\end{bmatrix}
= \delta_i v_i,
\]
so that \((\delta_i, v_i)\) is a right-eigenpair for \(P\). 

**Lemma 3.2.** If
\[
M = M_n := \begin{bmatrix}
1 & e^T \\
e & -I
\end{bmatrix} \in M_n(\mathbb{R}), \quad n \geq 2,
\]
then
\[
M^{-1} = M_n^{-1} = \frac{1}{n} \begin{bmatrix}
1 & e^T
\end{bmatrix} \begin{bmatrix}
e & J - nI
\end{bmatrix}.
\]

**Proof.** Clearly,
\[
nM M^{-1} = \begin{bmatrix}
1 & e^T \\
e & -I
\end{bmatrix} \cdot \begin{bmatrix}
1 & e^T \\
e & J - nI
\end{bmatrix} = \begin{bmatrix}
n & e^T + e^T (J - nI) \\
0 & nI
\end{bmatrix},
\]
but \(e^T + e^T (J - nI) = e^T + (n-1)e^T - ne^T = 0\); dividing through by \(n\) establishes the result.
Theorem 3.3 (Suleimanova [13]). Every Suleimanova spectrum is realizable.

Proof. Let $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ be a Suleimanova spectrum and assume, without loss of generality, that $\lambda_1 \geq 0 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If $\lambda := [\lambda_1 \lambda_2 \cdots \lambda_n]^T \in \mathbb{R}^n$, then, following Lemma 3.2, the solution $x$ of the linear system

$$
\begin{align*}
    x_1 + x_2 + \cdots + x_n &= \lambda_1 \\
    x_1 - x_2 &= \lambda_2 \\
    &\vdots \\
    x_1 - x_n &= \lambda_n
\end{align*}
$$

is given by

$$
x = M^{-1}\lambda = \frac{1}{n} \begin{bmatrix}
    s_1(\sigma) \\
    s_1(\sigma) - n\lambda_2 \\
    \vdots \\
    s_1(\sigma) - n\lambda_n
\end{bmatrix},
$$

which is clearly nonnegative. Following Lemma 3.1, the nonnegative matrix $P_x$ realizes $\sigma$. $\blacksquare$

Example 3.4. If $\sigma = \{10, -1, -2, -3\}$, then $\sigma$ is realizable by

$$
\begin{bmatrix}
    1 & 2 & 3 & 4 \\
    2 & 1 & 3 & 4 \\
    3 & 2 & 1 & 4 \\
    4 & 2 & 3 & 1
\end{bmatrix}.
$$

Corollary 3.5. If $\sigma = \{\lambda_1, -\lambda_2, \ldots, -\lambda_n\}$ is a Suleimanova spectrum such that $s_1(\sigma) = 0$ and $\lambda_1 > 0$, then the $n$-by-$n$ nonnegative matrix

$$
P := \begin{bmatrix}
    0 & \lambda_2 & \cdots & \lambda_i & \cdots & \lambda_n \\
    \lambda_2 & 0 & \cdots & \lambda_i & \cdots & \lambda_n \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    \lambda_i & \lambda_2 & \cdots & 0 & \cdots & \lambda_n \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    \lambda_n & \lambda_2 & \cdots & \lambda_i & \cdots & 0
\end{bmatrix}
$$

realizes $\sigma$.

Example 3.6. If $\sigma = \{6, -1, -2, -3\}$, then $\sigma$ is realizable by

$$
\begin{bmatrix}
    0 & 1 & 2 & 3 \\
    1 & 0 & 2 & 3 \\
    2 & 1 & 0 & 3 \\
    3 & 1 & 2 & 0
\end{bmatrix}.$$
4. Connection to the RNIEP. It is well-known that for $1 \leq n \leq 4$, conditions \((1.1)\) and \((1.2)\) are also sufficient for realizability (see, e.g., [6,7]). In this section, we strengthen this result by demonstrating that the realizing matrix can be taken to be permutative or a direct sum of permutative matrices.

**Theorem 4.1.** If $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ and $1 \leq n \leq 4$, then $\sigma$ is realizable if and only if $\sigma$ satisfies \((1.1)\) and \((1.2)\). Furthermore, the realizing matrix can be taken to be permutative or a direct sum of permutative matrices.

**Proof.** Without loss of generality, assume that $\rho(\sigma) = 1$.

The case when $n = 1$ is trivial, but it is worth mentioning that $\sigma = \{1\}$ is realized by the permutative matrix $[1]$.

If $\sigma = \{1, \lambda\}$, $-1 \leq \lambda \leq 1$, then the permutative matrix

\[
\frac{1}{2} \begin{bmatrix}
1 + \lambda & 1 - \lambda \\
1 - \lambda & 1 + \lambda
\end{bmatrix}
\]

realizes $\sigma$.

As established in [6], if $\sigma = \{1, \mu, \lambda\}$, where $-1 \leq \mu, \lambda \leq 1$, then the matrix

\[
\begin{bmatrix}
\frac{(1 + \lambda)/2}{(1 - \lambda)/2} & \frac{(1 - \lambda)/2}{(1 + \lambda)/2} & 0 \\
0 & 0 & \mu
\end{bmatrix}
\]

realizes $\sigma$ when $1 \geq \mu \geq \lambda \geq 0$ or $1 \geq \mu \geq 0 > \lambda$. Notice that this matrix is a direct sum of permutative matrices. If $0 > \mu \geq \lambda$, then, following Theorem 3.3, $\sigma$ is realizable by a permutative matrix.

When $n = 4$, all realizable spectra can be realized by matrices of the form

\[
\begin{bmatrix}
a + b & a - b & 0 & 0 \\
a - b & a + b & 0 & 0 \\
0 & 0 & c + d & c - d \\
0 & 0 & c - d & c + d
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
a & b & c & d \\
b & a & d & c \\
c & d & a & b \\
d & c & b & a
\end{bmatrix}
\]

(for full details, see [6, pp. 10–11]).

5. Concluding remarks. In [4], Fiedler posed the symmetric nonnegative inverse eigenvalue problem (SNIEP), which requires the realizing matrix to be symmetric. Obviously, if $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ is a solution to the SNIEP, then it is a solution to the RNIEP. In [5], Johnson, Laffey, and Loewy showed that the RNIEP strictly
contains the SNIEP when \( n \geq 5 \). It is in the spirit of this problem that we pose the following.

**Problem 5.1.** Can all realizable real spectra be realized by a permutative matrix or by a direct sum of permutative matrices?

At this point there is no evidence that suggests an affirmative answer to Problem 5.1; however, a negative answer could be just as difficult: one possibility, communicated to me by R. Loewy, is to find an extreme nonnegative matrix \([8]\) with a real spectrum that cannot be realized by a permutative matrix, or a direct sum of permutative matrices.

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**REFERENCES**


