Realizing Suleimanova-type Spectra via Permutative Matrices

Pietro Paparella

University of Washington - Bothell Campus, pietrop@uw.edu

Follow this and additional works at: http://repository.uwyo.edu/ela

Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.3101

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in Electronic Journal of Linear Algebra by an authorized editor of Wyoming Scholars Repository. For more information, please contact scholcom@uwyo.edu.
REALIZING SULEIMANOVA SPECTRA VIA
PERMUTATIVE MATRICES∗

PIETRO PAPARELLA†

Abstract. A permutative matrix is a square matrix such that every row is a permutation of the first row. A constructive version of a result attributed to Suleimanova is given via permutative matrices. A well-known result is strengthened by showing that all realizable spectra containing at most four elements can be realized by a permutative matrix or by a direct sum of permutative matrices. The paper concludes by posing a problem.

Key words. Suleimanova spectrum, Permutative matrix, Real nonnegative inverse eigenvalue problem.

AMS subject classifications. 15A18, 15A29, 15B99.

1. Introduction. Introduced by Suleimanova in [13], the longstanding real non-negative inverse eigenvalue problem (RNIEP) is to determine necessary and sufficient conditions on a set \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R} \) so that \( \sigma \) is the spectrum an \( n \)-by-\( n \) entrywise nonnegative matrix.

If \( A \) is an \( n \)-by-\( n \) nonnegative matrix with spectrum \( \sigma \), then \( \sigma \) said to be realizable and the matrix \( A \) is called a realizing matrix for \( \sigma \). It is well-known that if \( \sigma \) is realizable, then

\[
s_k(\sigma) := \sum_{i=1}^{n} \lambda_i^k \geq 0, \quad \forall \, k \in \mathbb{N}
\]

\[
\rho(\sigma) := \max_{1 \leq i \leq n} |\lambda_i| \in \sigma.
\]

For additional background and results, see, e.g., [2, 9] and references therein.

A set \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R} \) is called a Suleimanova spectrum if \( s_1(\sigma) \geq 0 \) and \( \sigma \) contains exactly one positive element. Suleimanova [13] announced (and loosely proved) that every such spectrum is realizable. Fiedler [3] showed that every Suleimanova spectrum is symmetrically realizable (i.e., realizable by a symmetric nonnegative matrix), however, his proof is by induction and does not explicitly yield a realizing

∗Received by the editors on September 2, 2015. Accepted for publication on December 16, 2015. Handling Editor: Raphael Loewy.
†Division of Engineering and Mathematics, University of Washington Bothell, Bothell, WA 98011-8246, USA (pietrop@uw.edu).
matrix for all orders. In [6], Johnson and Paparella provide a constructive version of Fiedler’s result for Hadamard orders.

Friedland [4] and Perfect [10] proved Suleimanova’s result via companion matrices (for other proofs, see references in [4]). In particular, the coefficients $c_0, c_1, \ldots, c_{n-1}$ of the polynomial $p(t) := \prod_{k=1}^{n}(t - \lambda_k) = t^n + \sum_{k=0}^{n-1} c_k t^k$ are nonpositive so that the companion matrix of $p$ is nonnegative. As noted in [11, p. 1380], the construction of the companion matrix of $p$ requires evaluating the elementary symmetric functions at $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a computation with $O(2^n)$ complexity.

The computation of a realizing matrix for a realizable spectrum is of obvious interest for numerical purposes, but for many known theoretical results, a realizing matrix is not readily available. Indeed, according to Chu:

Very few of these theoretical results are ready for implementation to actually compute [the realizing] matrix. The most constructive result we have seen is the sufficient condition studied by Soules [12]. But the condition there is still limited because the construction depends on the specification of the Perron vector – in particular, the components of the Perron eigenvector need to satisfy certain inequalities in order for the construction to work. [1, p. 18].

In this work, we provide a constructive version of Suleimanova’s result via permutative matrices. The paper is organized as follows: Section 2 contains notation and definitions; Section 3 contains the main results; in Section 4 we show that if $\sigma = \{\lambda_1, \ldots, \lambda_n\}, n \leq 4$, satisfies (1.1) and (1.2), then $\sigma$ is realizable by a permutative matrix or by a direct sum of permutative matrices; and we conclude by posing a problem in Section 5.

2. Notation. The set of $m$-by-$n$ matrices with entries from a field $\mathbb{F}$ (in this paper, $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$) is denoted by $M_{m,n}(\mathbb{F})$ (when $m = n$, $M_{m,n}(\mathbb{F})$ is abbreviated to $M_n(\mathbb{F})$). For $A = [a_{ij}] \in M_n(\mathbb{C})$, $\sigma(A)$ denotes the spectrum of $A$.

The set of $n$-by-$1$ column vectors is identified with the set of all $n$-tuples with entries in $\mathbb{F}$ and thus denoted by $\mathbb{F}^n$. Given $x \in \mathbb{F}^n$, $x_i$ denotes the $i^{th}$ entry of $x$.

For the following, the size of each object will be clear from the context in which it appears:

- $I$ denotes the identity matrix;
- $e$ denotes the all-ones vector; and
- $J$ denotes the all-ones matrix, i.e., $J = ee^T$. 


308

P. Paparella

Definition 2.1. For $x \in \mathbb{C}^n$ and permutation matrices $P_2, \ldots, P_n \in M_n(\mathbb{R})$, a permutative matrix is any matrix of the form

$$
\begin{bmatrix}
    x^T \\
    (P_2x)^T \\
    \vdots \\
    (P_nx)^T
\end{bmatrix} \in M_n(\mathbb{C}).
$$

According to Definition 2.1, all one-by-one matrices are considered permutative.

3. Main results. We begin with the following lemmas.

Lemma 3.1. For $x \in \mathbb{C}^n$, let

$$
P = P_x = \begin{bmatrix}
    1 & 2 & \cdots & i & \cdots & n \\
    x_1 & x_2 & \cdots & x_i & \cdots & x_n \\
    x_2 & x_1 & \cdots & x_i & \cdots & x_n \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    i & x_i & x_2 & \cdots & x_1 & \cdots & x_n \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
    n & x_n & x_2 & \cdots & x_i & \cdots & x_1 
\end{bmatrix} = \begin{bmatrix}
    x^T \\
    (P_\alpha x)^T \\
    \vdots \\
    (P_n x)^T
\end{bmatrix},
$$

where $P_\alpha$ is the permutation matrix corresponding to the permutation $\alpha_i$ defined by $\alpha_i(x) = (1, i), i = 2, \ldots, n$. Then $\sigma(P) = \{s, \delta_2, \ldots, \delta_n\}$, where $s := \sum_{i=1}^n x_i$ and $\delta_i := x_1 - x_i, i = 2, \ldots, n$.

Proof. Since every row sum of $P$ is $s$, it follows that $Pe = se$, i.e., $s \in \sigma(P)$.

Since

$$
P - \delta_i I = \begin{bmatrix}
    1 & 2 & \cdots & i & \cdots & n \\
    x_i & x_2 & \cdots & x_i & \cdots & x_n \\
    x_2 & x_i & \cdots & x_i & \cdots & x_n \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    i & x_i & x_2 & \cdots & x_i & \cdots & x_n \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
    n & x_n & x_2 & \cdots & x_i & \cdots & x_1 
\end{bmatrix},
$$

it follows that the homogeneous linear system $(P - \delta_i I)x = 0$ has a nontrivial solution (notice that the first and $i$th rows of $P - \delta_i I$ are identical). Thus, $\delta_i \in \sigma(P)$.

---

Terminology due to Charles R. Johnson.
Moreover, if

\[ \begin{bmatrix}
  1 & x_i \\
  \vdots & \vdots \\
  i-1 & x_i \\
  i & x_1 - s \\
  i+1 & x_i \\
  \vdots & \vdots \\
  n & x_i \\
\end{bmatrix}, \quad i = 2, \ldots, n \]

then

\[ P \begin{bmatrix}
  1 & x_i(s - x_i) + x_i(x_1 - s) \\
  \vdots & \vdots \\
  i-1 & x_i(s - x_i) + x_i(x_1 - s) \\
  i & x_i(x_1 - x_i) + x_i(x_1 - s) \\
  i+1 & x_i(s - x_i) + x_i(x_1 - s) \\
  \vdots & \vdots \\
  n & x_i(s - x_i) + x_i(x_1 - s) \\
\end{bmatrix} = (x_1 - x_i) \begin{bmatrix}
  1 & x_1 - s \\
  \vdots & \vdots \\
  i & x_i \\
\end{bmatrix} = \delta_i \begin{bmatrix}
  1 & x_1 - s \\
  \vdots & \vdots \\
  i & x_i \\
\end{bmatrix}, \]

so that \((\delta_i, v_i)\) is a right-eigenpair for \(P\).

**Lemma 3.2.** If

\[ M = M_n := \begin{bmatrix}
  1 & e^\top \\
  e & -I \\
\end{bmatrix} \in M_n(\mathbb{R}), \quad n \geq 2, \]

then

\[ M^{-1} = M_n^{-1} = \frac{1}{n} \begin{bmatrix}
  1 & e^\top \\
  e & J - nI \\
\end{bmatrix}. \]

**Proof.** Clearly,

\[ nM M^{-1} = \begin{bmatrix}
  1 & e^\top \\
  e & -I \\
\end{bmatrix} \cdot \begin{bmatrix}
  1 & e^\top \\
  e & J - nI \\
\end{bmatrix} = \begin{bmatrix}
  n & e^\top + e^\top (J - nI) \\
  0 & nI \\
\end{bmatrix}, \]

but \(e^\top + e^\top (J - nI) = e^\top + (n-1)e^\top - ne^\top = 0\); dividing through by \(n\) establishes the result.
Theorem 3.3 (Suleimanova [13]). Every Suleimanova spectrum is realizable.

Proof. Let \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \) be a Suleimanova spectrum and assume, without loss of generality, that \( \lambda_1 \geq 0 \geq \lambda_2 \geq \cdots \geq \lambda_n \). If \( \lambda := [\lambda_1 \lambda_2 \cdots \lambda_n]^T \in \mathbb{R}^n \), then, following Lemma 3.2, the solution \( x \) of the linear system
\[
\begin{align*}
x_1 + x_2 + \cdots + x_n &= \lambda_1 \\
x_1 - x_2 &= \lambda_2 \\
&\vdots \\
x_1 - x_n &= \lambda_n
\end{align*}
\]
is given by
\[
x = M^{-1} \lambda = \frac{1}{n} \begin{bmatrix} s_1(\sigma) \\ s_1(\sigma) - n\lambda_2 \\ \vdots \\ s_1(\sigma) - n\lambda_n \end{bmatrix},
\]
which is clearly nonnegative. Following Lemma 3.1, the nonnegative matrix \( P \) realizes \( \sigma \).

Example 3.4. If \( \sigma = \{10, -1, -2, -3\} \), then \( \sigma \) is realizable by
\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 3 & 4 \\
3 & 2 & 1 & 4 \\
4 & 2 & 3 & 1
\end{bmatrix}
\]

Corollary 3.5. If \( \sigma = \{\lambda_1, -\lambda_2, \ldots, -\lambda_n\} \) is a Suleimanova spectrum such that \( s_1(\sigma) = 0 \) and \( \lambda_1 > 0 \), then the \( n \times n \) nonnegative matrix
\[
P := \begin{bmatrix}
0 & \lambda_2 & \cdots & \lambda_i & \cdots & \lambda_n \\
\lambda_2 & 0 & \cdots & \lambda_i & \cdots & \lambda_n \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\lambda_i & \lambda_2 & \cdots & 0 & \cdots & \lambda_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_n & \lambda_2 & \cdots & \lambda_i & \cdots & 0
\end{bmatrix}
\]
realizes \( \sigma \).

Example 3.6. If \( \sigma = \{6, -1, -2, -3\} \), then \( \sigma \) is realizable by
\[
\begin{bmatrix}
0 & 1 & 2 & 3 \\
1 & 0 & 2 & 3 \\
2 & 1 & 0 & 3 \\
3 & 1 & 2 & 0
\end{bmatrix}
\]
4. Connection to the RNIEP. It is well-known that for $1 \leq n \leq 4$, conditions (1.1) and (1.2) are also sufficient for realizability (see, e.g., [6, 7]). In this section, we strengthen this result by demonstrating that the realizing matrix can be taken to be permutative or a direct sum of permutative matrices.

**Theorem 4.1.** If $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ and $1 \leq n \leq 4$, then $\sigma$ is realizable if and only if $\sigma$ satisfies (1.1) and (1.2). Furthermore, the realizing matrix can be taken to be permutative or a direct sum of permutative matrices.

**Proof.** Without loss of generality, assume that $\rho(\sigma) = 1$.

The case when $n = 1$ is trivial, but it is worth mentioning that $\sigma = \{1\}$ is realized by the permutative matrix $[1]$.

If $\sigma = \{1, \lambda\}$, $-1 \leq \lambda \leq 1$, then the permutative matrix

$$
\begin{bmatrix}
\frac{1}{2} [1 + \lambda & 1 - \lambda] \\
\frac{1}{2} [1 - \lambda & 1 + \lambda]
\end{bmatrix}
$$

realizes $\sigma$.

As established in [6], if $\sigma = \{1, \mu, \lambda\}$, where $-1 \leq \mu, \lambda \leq 1$, then the matrix

$$
\begin{bmatrix}
\frac{(1 + \lambda)/2}{(1 - \lambda)/2} & \frac{(1 - \lambda)/2}{(1 + \lambda)/2} & 0 \\
0 & 0 & \mu
\end{bmatrix}
$$

realizes $\sigma$ when $1 \geq \mu \geq \lambda \geq 0$ or $1 \geq \mu > 0 \geq \lambda$. Notice that this matrix is a direct sum of permutative matrices. If $0 > \mu \geq \lambda$, then, following Theorem 3.3, $\sigma$ is realizable by a permutative matrix.

When $n = 4$, all realizable spectra can be realized by matrices of the form

$$
\begin{bmatrix}
a + b & a - b & 0 & 0 \\
a - b & a + b & 0 & 0 \\
0 & 0 & c + d & c - d \\
0 & 0 & c - d & c + d
\end{bmatrix}
$$

or

$$
\begin{bmatrix}
a & b & c & d \\
b & a & d & c \\
c & d & a & b \\
d & c & b & a
\end{bmatrix}
$$

(for full details, see [6] pp. 10–11).

5. Concluding remarks. In [4], Fiedler posed the symmetric nonnegative inverse eigenvalue problem (SNIEP), which requires the realizing matrix to be symmetric. Obviously, if $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ is a solution to the SNIEP, then it is a solution to the RNIEP. In [5], Johnson, Laffey, and Loewy showed that the RNIEP strictly
contains the SNIEP when \( n \geq 5 \). It is in the spirit of this problem that we pose the following.

**Problem 5.1.** Can all realizable real spectra be realized by a permutative matrix or by a direct sum of permutative matrices?

At this point there is no evidence that suggests an affirmative answer to Problem 5.1; however, a negative answer could be just as difficult: one possibility, communicated to me by R. Loewy, is to find an extreme nonnegative matrix \([8]\) with a real spectrum that can not be realized by a permutative matrix, or a direct sum of permutative matrices.

**Acknowledgment.** I wish to thank the anonymous referee and Raphael Loewy for their comments on improving the first-draft.

**REFERENCES**