Orthogonal bases of Brauer symmetry classes of
tensors for groups having cyclic support on non-
linear Brauer characters

Mahdi Hormozi
Department of Mathematical Sciences, Division of Mathematics, Chalmers University of Technology and University of
Gothenburg, Gothenburg 41296, Sweden, hormozi@chalmers.se

Kijti Rodtes
Department of Mathematics, Faculty of Science, Naresuan University, kijtir@nu.ac.th

Follow this and additional works at: http://repository.uwyo.edu/ela

Part of the Algebra Commons

Recommended Citation
Hormozi, Mahdi and Rodtes, Kijti. (2016), "Orthogonal bases of Brauer symmetry classes of tensors for groups having cyclic support
DOI: https://doi.org/10.13001/1081-3810.3155
ORTHOGONAL BASES OF BRAUER SYMMETRY CLASSES OF TENSORS FOR GROUPS HAVING CYCLIC SUPPORT ON NON-LINEAR BRAUER CHARACTERS

MAHDI HORMOZI† AND KIJTI RODTES‡

Abstract. This paper provides some properties of Brauer symmetry classes of tensors. A dimension formula is derived for the orbital subspaces in the Brauer symmetry classes of tensors corresponding to the irreducible Brauer characters of the groups whose non-linear Brauer characters have support being a cyclic group. Using the derived formula, necessary and sufficient condition are investigated for the existence of an o-basis of dicyclic groups, semi-dihedral groups, and also those things are reinvestigated on dihedral groups. Some criteria for the non-vanishing elements in the Brauer symmetry classes of tensors associated to those groups are also included.

Key words. Brauer symmetry classes of tensors, Orthogonal basis, Semi-dihedral groups, Dicyclic groups.

AMS subject classifications. 20C30, 15A69.

1. Introduction. During the past decades, there are many papers devoted to study symmetry classes of tensors, see, for example, [1]–[9]. One of the active research topics is the investigation of a special basis (called an o-basis) for the classes. This basis consists of decomposable symmetrized tensors that are images of the symmetrizer using an irreducible character of a given group. In [10], Randall R. Holmes and A. Kodithuwakku studied symmetry classes of tensors using an irreducible Brauer character of the dihedral group instead of an ordinary irreducible character and gave necessary and sufficient conditions for the existence of an o-basis. A classical method to provide the conditions applies the dimension of the orbital subspaces in order to find an o-basis for each orbit separately. A main tool for computing the dimension of symmetry classes using ordinary characters is the Freese’s theorem [9]. Unfortunately, the symmetrizer using Brauer characters is not (in general) idempotent, so the Freese’s theorem can not be applied directly. However, for the case of non-linear Brauer characters of dihedral groups, the authors in [10] decomposed them into a
sum of ordinary characters and used the generalized Freese’s theorem to bound the dimension.

One common property for all non-linear Brauer characters of dihedral groups is their vanishing outside some cyclic subgroups. Many finite groups, including dicyclic groups and semi-dihedral groups, satisfy this property. In this paper, we investigate the existence of an $o$-basis of Brauer symmetry classes of tensors associated with the groups having the stated property. Some properties of symmetry classes of tensors symetrized using a complex value function are stated. For the non-linear Brauer characters, we decompose the orbital subspaces of Brauer symmetry classes of tensors into an orthogonal direct sum of smaller factors and then provide a dimension formula for each of them. The necessary and sufficient condition for the existence of an $o$-basis for dicyclic groups, semi-dihedral groups and dihedral groups are investigated and reinvestigated as an application of the formula. Some criteria for the non-vanishing elements in the Brauer symmetry classes of tensors associated to these groups are also included.

2. Preliminaries. Let $G$ be a subgroup of the full symmetric group $S_m$ and $p$ be a fixed prime number. An element of $G$ is $p$-regular if its order is not divisible by $p$. Denote by $\hat{G}$ the set of all $p$-regular elements of $G$. Let $\text{IBr}(G)$ denote the set of irreducible Brauer characters of $G$. A Brauer character is a certain function from $\hat{G}$ to $\mathbb{C}$ associated with an $FG$-module where $F$ is a suitably chosen field of characteristic $p$. The Brauer character is irreducible if the associated module is simple. A conjugacy class of $G$ consisting of $p$-regular elements is called a $p$-regular class. The number of irreducible Brauer characters of $G$ equals the number of $p$-regular classes of $G$. Let $\text{Irr}(G)$ denote the set of irreducible characters of $G$. (Unless preceded by the word Brauer, the word character always refers to an ordinary character.) If the order of $G$ is not divisible by $p$, then $\hat{G} = G$ and $\text{IBr}(G) = \text{Irr}(G)$. Let $S$ be a subset of $G$ containing the identity element $e$ and let $\phi : S \to \mathbb{C}$ be a fixed function. Statements below involving $\phi$ hold if $\phi$ is a character of $G$ (in which case $S = G$) and also if $\phi$ is a Brauer character of $G$ (in which case $S = \hat{G}$). During the last few years, many very interesting results on the topic of Brauer characters have been found (see e.g. [13] and [15–22]).

Let $V$ be a $k$-dimensional complex inner product space and $\{e_1, \ldots, e_k\}$ be an orthonormal basis of $V$. Let $\Gamma_k^m$ be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_m)$, with $1 \leq \alpha_i \leq k$. Define the action of $G$ on $\Gamma_k^m$ by

$$o\sigma = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(m)})$$

We denote by $G_\alpha$ the stabilizer subgroup of $\alpha$, i.e., $G_\alpha = \{\sigma \in G | o\sigma = \alpha\}$. The space $V^{\otimes m}$ is a left $CG$-module with the action given $\sigma e_\gamma = e_{\gamma\sigma^{-1}}$ ($\sigma \in G, \gamma \in \Gamma_k^m$) extended linearly. The inner product on $V$ induces an inner product on $V^{\otimes m}$ which
Orthogonal Bases of Brauer Symmetry Classes of Tensors for Groups Having Cyclic Support

is $G$-invariant and, with respect to this inner product, the set $\{ e_\alpha | \alpha \in \Gamma_{m_k}^m \}$ is an orthonormal basis for $V_{\otimes m_k}^\phi$, where $e_\alpha = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m}$.

The symmetrizer corresponding to $\phi$ and $S \subseteq G$ is the element $s_\phi$ of the group algebra $\mathbb{C}G$ given by

$$s_\phi = \frac{\phi(e)}{|S|} \sum_{\sigma \in S} \phi(\sigma)\sigma.$$  \hspace{1cm} (2.1)

Corresponding to $\phi$ and $\alpha \in \Gamma_{m_k}^m$, the standard (or decomposable) symmetrized tensor is

$$e_\phi^\alpha = s_\phi e_\alpha = \frac{\phi(e)}{|S|} \sum_{\sigma \in S} \phi(\sigma)e_{\alpha\sigma^{-1}}.$$  \hspace{1cm} (2.2)

The symmetry class of tensors associated with $\phi$ and $S \subseteq G$ is

$$V_\phi^S(G) = s_\phi V_{\otimes m_k} = \langle e_\phi^\alpha | \alpha \in \Gamma_{m_k}^m \rangle.$$

If $\phi$ is a Brauer character, we refer to $V_\phi^S(G)$ as a Brauer symmetry class of tensors.

The orbital subspace of $V_\phi^S(G)$ corresponding to $\alpha \in \Gamma_{m_k}^m$ is

$$V_\alpha^\phi(G) = \langle e_\phi^\alpha | \sigma \in G \rangle.$$

An o-basis of a subspace $W$ of $V_\phi^S(G)$ is an orthogonal basis of $W$ of the form $\{e_\phi^{\alpha_1}, \ldots, e_\phi^{\alpha_t}\}$ for some $\alpha_i \in \Gamma_{m_k}^m$. By convention, the empty set is regarded as an o-basis of the zero subspace of $V_\phi^S(G)$. Let $\Delta = \Delta_G$ be a set of representatives of the orbits of $\Gamma_{m_k}^m$ under the action of $G$.

The following critical theorem is used to reduce the task of investigation on the existence of an o-basis.

**Theorem 2.1.** We have an orthogonal sum decomposition

$$V_\phi^S(G) = \sum_{\alpha \in \Delta} V_\alpha^\phi(G).$$

**Proof.** See [10, Thm. 1.1]. \qed

The induced inner product on $V_\phi^S(G)$ can be calculated via the formula below, which is an adaptation from the Theorem 1.2 in [10].

**Theorem 2.2.** For every $\alpha \in \Gamma_{m_k}^m$ and $\sigma_1, \sigma_2 \in G$, we have

$$\langle e_\phi^{\alpha\sigma_1}, e_\phi^{\alpha\sigma_2} \rangle = \frac{\phi(e)^2}{|S|^2} \sum_{\mu \in S} \sum_{\tau \in \sigma_1^{-1}\alpha_1^{-1}\tau} \phi(\mu)\phi(\mu_2^{-1}\tau\sigma_2).$$  \hspace{1cm} (2.3)
Proof. For $\alpha \in \Gamma^m_k$ and $\sigma_1, \sigma_2 \in G$, we have
\[
\langle e^{\phi} \alpha \sigma_1, e^{\phi} \alpha \sigma_2 \rangle = |\phi(e)|^2 \sum_{\mu \in S} \sum_{\rho \in S} \phi(\mu) \overline{\phi(\rho)} \langle e^{\phi} \alpha \mu^{-1}, e^{\phi} \alpha \rho^{-1} \rangle
\]
\[
= |\phi(e)|^2 \sum_{\mu \in S} \sum_{\rho \in S} \phi(\mu) \overline{\phi(\rho)} \langle e^{\phi} \alpha \mu^{-1} \rho \tau^{-1}, e^{\phi} \alpha \tau^{-1} \rangle
\]
where $\tau = \sigma_1 \mu^{-1} \rho \tau^{-1}$.

The following is an immediate consequence of Theorem 2.2.

**Corollary 2.3.** Let $\sigma_1, \sigma_2 \in G, S \subseteq G$ and $\phi = \psi |_S$, where $\psi$ is a linear character of $G$. If $G_\alpha = \{e\}$ and $A = \{\mu \in S \mid e \in \sigma_1 \mu^{-1} S \sigma_2^{-1}\} \neq \emptyset$, then
\[
\langle e^{\phi} \alpha \sigma_1, e^{\phi} \alpha \sigma_2 \rangle \neq 0.
\]

In the following sections, we also need the lemma and propositions below.

**Lemma 2.4.** Assume that $S$ is closed under conjugation by elements of $G$ and that $\phi$ is constant on the conjugacy classes of $G$. For each $\alpha \in \Gamma^m_k$ and $\sigma \in G$, we have $\sigma e^{\phi} \alpha = e^{\phi} \alpha \sigma^{-1}$.

**Proof.** See [10, Lem. 1.3].

As an immediate consequence of this lemma, we have the following proposition.

**Proposition 2.5.** Let $\phi : S \rightarrow \mathbb{C}$ be a fixed function equipped the assumption of Lemma 2.4. If $B = \{e^{\phi} \alpha_1, e^{\phi} \alpha_2, \ldots, e^{\phi} \alpha_k\}$ is an o-basis of $V^{\phi}_\alpha(G)$, then, for each $g \in G$,
\[
g \cdot B = \{e^{\phi} \alpha_1 g^{-1}, e^{\phi} \alpha_2 g^{-1}, \ldots, e^{\phi} \alpha_k g^{-1}\}
\]
is also an o-basis of $V^{\phi}_\alpha(G)$.

**Proposition 2.6.** Let $\phi : S \rightarrow \mathbb{C}$ be a fixed function. Also, let $C$ contained in $S$ be a subgroup of $G$. If $G_\gamma = \{e\}$ and $\phi(s) = 0$ for all $s \in S - C$, then
\[
V^{\phi}_\gamma(G) = \langle e^{\phi} \gamma g \mid g \in C \rangle \oplus \langle e^{\phi} \gamma g \mid g \in G - C \rangle.
\]
**Proof.** If we choose \( \sigma_1 \in C \) and \( \sigma_2 \in G \setminus C \) in (2.3), we get nonzero term only if \( \mu \in C \) and \( \mu \sigma_1^{-1} \sigma_2 \in C \), which is impossible, since \( C \) is a group. Thus, the two spaces are orthogonal.

**Proposition 2.7.** Let \( S \) be a subgroup of \( G \) and \( \phi : S \rightarrow C \) be a nonzero constant function on \( S \). Then, for each \( \alpha \in \Gamma_m \dim V \),

\[
V^\phi_\alpha(G) = \langle e^\phi_{\alpha \sigma} | \sigma \in G \rangle
\]

has an \( o \)-basis and so does \( V_\phi(G) \).

**Proof.** Suppose \( \phi(s) = c \in C \) for all \( s \in S \). Since \( S \) is a group and by Theorem 2.2 we have that, for \( \sigma, \tau \in G \),

\[
\langle e^\phi_{\alpha \sigma}, e^\phi_{\alpha \tau} \rangle = \frac{|c|^2}{|S|^2} \sum_{\mu \in S} \sum_{\delta \in \mu^{-1}S^{-1} \cap G_\alpha} |e|^2
\]

\[
= \frac{|c|^4}{|S|^2} \sum_{\mu \in S} |\sigma \tau^{-1} \cap G_\alpha|
\]

\[
= \frac{|c|^4 |\sigma \tau^{-1} \cap G_\alpha|}{|S|}.
\]

We have \( G_\alpha \cap \sigma \tau^{-1} = \emptyset \) or \( G_\alpha \cap \sigma \tau^{-1} \neq \emptyset \), for each \( \sigma, \tau \in G \). For the latter case, we have \( \sigma \mu \tau^{-1} \in G_\alpha \), for some \( \mu \in S \). Thus, for each \( b \in S \),

\[
\alpha \sigma g = \alpha (\sigma \mu \tau^{-1}) (\tau \mu^{-1} b) = \alpha \tau g,
\]

for some \( g = \mu^{-1} b \in S \). Hence, \( \{ \alpha \sigma b | b \in S \} = \{ \alpha \tau b | b \in S \} \). Since \( S \) is a group and \( \phi(s) = c \) for all \( s \in S \), we have

\[
e^\phi_{\alpha \sigma} = \frac{c^2}{|S|} \sum_{s \in S} e_{s \sigma} = e^\phi_{\alpha \tau}.
\]

This implies that, for \( \sigma, \tau \in G \), \( e^\phi_{\alpha \sigma} = e^\phi_{\alpha \tau} \) or \( \langle e^\phi_{\alpha \sigma}, e^\phi_{\alpha \tau} \rangle = 0 \), which yields that \( V^\phi_\alpha(G) \) has an \( o \)-basis and by Theorem 2.1 we complete the proof.

**3. Dimension formula.** In this section, we let \( G \) be a finite group, \( S \subseteq G \) and \( C \) be a subgroup of \( G \) contained in \( S \). Let \( \phi : G \rightarrow C \) be a function such that \( \phi(\sigma) \neq 0 \) for each \( \sigma \in C \) but \( \phi(S \setminus C) = 0 \). Thus, under this assumption, the induced inner product (2.3) becomes

\[
\langle e^\phi_{\alpha \sigma_1}, e^\phi_{\alpha \sigma_2} \rangle = \frac{\phi(e)^2}{|S|^2} \sum_{\mu \in C} \sum_{\tau \in C} \phi(\mu) \phi(\mu \sigma_1^{-1} \tau \sigma_2) \tag{3.1}
\]
for every $\alpha \in \Gamma^n_k$ and $\sigma_1, \sigma_2 \in G$. If $\sigma_1 C \sigma_2^{-1} \cap G_\alpha = \emptyset$, then $\langle e^\phi_{\alpha \sigma_1}, e^\phi_{\alpha \sigma_2} \rangle = 0$. This motivates us to define a relation on $G$: for each $\alpha \in \Delta$,

$$\sigma_1 \sim_\alpha \sigma_2 \iff \sigma_1 \in G_\alpha \sigma_2 C$$

(3.2)

for all $\sigma_1, \sigma_2 \in G$. It is not hard to check that $\sim_\alpha$, for each $\alpha \in \Delta$, is an equivalent relation.

Now, we set $[\sigma]$ as the equivalent class containing $\sigma$, $R^G_\alpha$ the set of representative of $G/\sim_\alpha$ and $V^\phi_\alpha([\sigma]) := \langle e^\phi_{\sigma g} | g \in [\sigma] \rangle$. It is clear that $V^\phi_\alpha([\sigma])$ is a subspace of $V^\phi_\alpha(G)$.

**Lemma 3.1.** The space $V^\phi_\alpha(G)$ has an o-basis if and only if for each $\sigma \in R^G_\alpha$, $V^\phi_\alpha([\sigma])$ has an o-basis.

**Proof.** Suppose that $V^\phi_\alpha([\sigma])$ has an o-basis, for each $\sigma \in R^G_\alpha$. To show that the space $V^\phi_\alpha(G)$ has an o-basis, it suffices to prove that $V^\phi_\alpha([\sigma])$’s are orthogonal. Now, let $\sigma_1, \sigma_2 \in G$ and $\alpha \in \Delta$. If $\sigma_1 C \sigma_2^{-1} \cap G_\alpha = \emptyset$, then $\sigma_1 \in G_\alpha \sigma_2 C$. Hence, if $\sigma_1 \sim_\alpha \sigma_2$, then $\sigma_1 C \sigma_2^{-1} \cap G_\alpha = \emptyset$. In other words, if $[\sigma_1] \neq [\sigma_2]$, then $\langle e^\phi_{\sigma_1}, e^\phi_{\sigma_2} \rangle = 0$. The other implication is clear. \[\square\]

For the following propositions, denote $\langle e^\phi_{\gamma g} | g \in C \rangle$ by $V^\phi_\gamma(C)$.

**Proposition 3.2.** The space $V^\phi_\alpha(G)$ has an o-basis if and only if for each $\gamma \in \Delta$, $V^\phi_\gamma(C)$ has an o-basis.

**Proof.** For each $[\sigma] \in G/\sim_\alpha$, we have that

$$V^\phi_\alpha([\sigma]) = \langle e^\phi_{\sigma g} | g \in [\sigma] \rangle = \langle e^\phi_{\sigma g} | g \in \sigma C \rangle = \langle \alpha \sigma g | g \in C \rangle$$

By Lemma 3.1 and (2.1), we finish the proof. \[\square\]

To determine the dimension of $V^\phi_\gamma(C)$, for each $\gamma \in \Delta$, we introduce a relation $\sim^*_\gamma$ on $C$ by: for each $\sigma_1, \sigma_2 \in C$,

$$\sigma_1 \sim^*_\gamma \sigma_2 \iff \sigma_1 \sigma_2^{-1} \in G_\gamma.$$  

(3.3)

It is obvious that $\sim^*_\gamma$ is an equivalent relation. Now, we have:

**Proposition 3.3.** If $C/\sim^*_\gamma = \{[\sigma_1], [\sigma_2], \ldots, [\sigma_t]\}$, then $\dim(V^\phi_\gamma(C)) = \text{rank}(M_\gamma)$, where $(M_\gamma)_{ij} := \sum_{k \in G_\gamma \sigma_j} \alpha(k \sigma_i)$ and $1 \leq i, j \leq t$.

**Proof.** For each $j \in \{1, 2, \ldots, t\}$ and $g_1, g_2 \in [\sigma_j]$, we have that $g_1 = c g_2$ for
some $c \in G_\gamma$. Thus,

\[
\begin{align*}
e^\phi_{\gamma g_1} &= \frac{\phi(e)}{|S|} \sum_{\sigma \in C} \phi(\sigma g_1) e_{\gamma \sigma^{-1}} \\
&= \frac{\phi(e)}{|S|} \sum_{\sigma \in C} \phi(\sigma g_2) e_{\gamma \sigma^{-1}} \\
&= \frac{\phi(e)}{|S|} \sum_{\tau \in C} \phi(\tau g_2) e_{\gamma \tau^{-1}} = e^\phi_{\gamma g_2}.
\end{align*}
\]

Hence, $V^\phi(C) = \{e^\phi_{\gamma g_i} | i = 1, 2, \ldots, t_\gamma\}$. Moreover, note that $e^\phi_{\gamma g_i} = e^\phi_{\gamma g_{i+1}}$ if $g_1, g_2 \in [\sigma_i]$. This yields

\[
e^\phi_{\gamma g} = \frac{\phi(e)}{|S|} \sum_{i=1}^{t_\gamma} \phi(\sigma g) e_{\gamma \sigma^{-1}},
\]

for each $g \in C$. However, $\sum_{\sigma \in [\sigma_i]} \phi(\sigma g) = \sum_{h \in C \cap G_\gamma} \phi(h \sigma g)$. So, we have

\[
e^\phi_{\gamma \sigma_j} = \frac{\phi(e)}{|S|} \sum_{i=1}^{t_\gamma} \left( \sum_{h \in C \cap G_\gamma} \phi(h \sigma_j) \right) e_{\gamma \sigma_j^{-1}}, \quad 1 \leq j \leq t_\gamma.
\]

The result follows by \((M_\gamma)_{ij} := \sum_{h \in C \cap G_\gamma} \phi(h \sigma_j) \) for $1 \leq i, j \leq t_\gamma$. 

In particular, as a special case of Proposition \ref{prop:dimformula}, i.e., if $C$ is a cyclic subgroup of $G$, we obtain a dimension formula for $V^\phi(C)$.

**Theorem 3.4.** Let $C = (\tau) \subseteq S$ be a cyclic subgroup of $G$ such that $C/ \sim^*_\gamma = \{[\tau], [\tau^2], \ldots, [\tau^{t_\gamma}]\}$. Denote $v_j = \sum_{h \in C \cap G_\gamma} \phi(h \tau^{j-1})$ and

\[
d_\gamma = \left| \left\{ s \in \{0, 1, 2, \ldots, t_\gamma - 1 \} \mid \sum_{j=0}^{t_\gamma-1} v_j e^{\frac{\pi i s j}{t_\gamma}} = 0 \right\} \right|.
\]

Then $t_\gamma = \frac{|C|}{|C \cap G_\gamma|}$ and $\dim(V^\phi_\gamma(C)) = t_\gamma - d_\gamma$.

**Proof.** Note that under the equivalent relation $\sim^*_\gamma$ with $C/ \sim^*_\gamma = \{[\tau], [\tau^2], \ldots, [\tau^{t_\gamma}]\}$, we have that $[\tau^k] = \{\sigma \in C \mid \sigma \tau^{-k} \in G_\gamma\} = \{h \tau^k \mid h \in C \cap G_\gamma\}$. So, $|[\tau^k]| = |C \cap G_\gamma|$ for all $k = 1, 2, \ldots, t_\gamma$, and hence,

\[
t_\gamma = |C/ \sim^*_\gamma| = \frac{|C|}{|C \cap G_\gamma|}.
\]

By rank nullity theorem and Proposition \ref{prop:dimformula}

\[
\dim(V^\phi_\gamma(C)) = \text{rank}(M_\gamma) = t_\gamma - \text{nullity}(M_\gamma) = t_\gamma - d_\gamma,
\]

where $d_\gamma := \text{nullity}(M_\gamma)$. 


To determine \( d_\gamma \), we observe that, if \( C \) is a cyclic subgroup of \( G \), then \( M_\gamma \) can be reduced to a circulant matrix \( M_\gamma^{\text{cir}} \) by swamping some columns of \( M_\gamma \). Precisely, \( M_\gamma^{\text{cir}} = (v_0, v_1, \ldots, v_{t_\gamma - 1}) \), where, for each \( j = 0, 1, \ldots, t_\gamma - 1 \),
\[
v_j = \sum_{h \in C \cap G_\gamma} \phi(h t_\gamma^{-j}).
\]

It is well known that (see e.g. [12]).
\[
\text{nullity}(M_\gamma^{\text{cir}}) = \text{deg}[\gcd(P_v(x), x^{t_\gamma} - 1)],
\]
where \( P_v(x) = \sum_{j=0}^{t_\gamma-1} v_j x^j \). Note that the set of all roots (over field \( \mathbb{C} \)) of \( x^{t_\gamma} - 1 \) is \( U := \{ e^{\frac{2\pi i s}{t_\gamma}} \mid 0 \leq s < t_\gamma \} \). Thus, common factors of \( P_v(x) \) and \( x^{t_\gamma} - 1 \) must have roots in \( U \) and hence,
\[
\text{deg}[\gcd(P_v(x), x^{t_\gamma} - 1)] = |\{ s \in \mathbb{Z} \mid 0 \leq s < t_\gamma \text{ and } P_v(e^{\frac{2\pi i s}{t_\gamma}}) = 0 \}| = |\{ s \in \mathbb{Z} \mid 0 \leq s < t_\gamma \text{ and } \sum_{j=0}^{t_\gamma-1} v_j e^{\frac{2\pi i j s}{t_\gamma}} = 0 \}|.
\]

Since the rank is invariant under column operations, \( d_\gamma = \text{nullity}(M_\gamma^{\text{cir}}) \) and thus the result follows.

4. Dicyclic group \( T_{4n} \). The dicyclic group \( T_{4n} \) is defined as follows:
\[
T_{4n} = \langle r, s \mid r^{2n} = e, r^n = s^2, s^{-1}rs = r^{-1} \rangle.
\]

Explicitly, all elements of the group \( T_{4n} \) may be given by \( T_{4n} = \{ r^i, sr^i \mid 0 \leq i < 2n \} \).

By the classical Cayley theorem, \( T_{4n} \) can be embedded in \( S_{4n} \). Precisely,
\[
\begin{align*}
  r &= ( \begin{array}{cccccc}
    1 & 2 & 3 & \cdots & 2n \\
    2n+1 & 2n+2 & 2n+3 & \cdots & 4n
  \end{array} ) ( \begin{array}{cccc}
    2n+1 & 2n+2 & 2n+3 & \cdots & 4n
  \end{array} ) \\
  s &= ( \begin{array}{cccccc}
    1 & 2n+1 & n+1 & 3n+1 \\
    2n+1 & n+1 & 3n+1 & 3n-1 & \cdots & 2n+3
  \end{array} ) ( \begin{array}{cccc}
    2n+1 & n+1 & 3n+1 & 3n-1 & \cdots & 2n+3
  \end{array} ) \\
&\quad ( \begin{array}{cccc}
    n & 3n+2 & 2n & 2n+2
  \end{array} ) \cdots ( \begin{array}{cccc}
    n-1 & 3n+3 & 2n-1 & 2n+3
  \end{array} )
\end{align*}
\]

\( T_{4n} \) has \( n + 3 \) conjugacy classes which are
\[
\{ e \}, \{ r^k, r^{2n-k} \}, 1 \leq k \leq n, \{ sr^k \mid 0 \leq k < n-1 \}, \{ sr^{2k+1} \mid 0 \leq k \leq n-1 \}
\]
and the ordinary irreducible character of \( T_{4n} \) are given by (see [11]):
Orthogonal Bases of Brauer Symmetry Classes of Tensors for Groups Having Cyclic Support

Table I: The character table for $T_{4n}$, when $n$ is even.

<table>
<thead>
<tr>
<th>Characters</th>
<th>$r^k (0 \leq k \leq n)$</th>
<th>$s$</th>
<th>$rs$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>$(-1)^k$</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>$(-1)^k$</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_j$, where $1 \leq j \leq n - 1$</td>
<td>$2 \cos \left( \frac{kj\pi}{n} \right)$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table II: The character table for $T_{4n}$, when $n$ is odd.

<table>
<thead>
<tr>
<th>Characters</th>
<th>$r^k (0 \leq k \leq n)$</th>
<th>$s$</th>
<th>$rs$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi'_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi'_1$</td>
<td>$(-1)^k$</td>
<td>i</td>
<td>-i</td>
</tr>
<tr>
<td>$\chi'_2$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi'_3$</td>
<td>$(-1)^k$</td>
<td>-i</td>
<td>i</td>
</tr>
<tr>
<td>$\psi'_j$, where $1 \leq j \leq n - 1$</td>
<td>$2 \cos \left( \frac{kj\pi}{n} \right)$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Write $2n = lp^t$ with $l$ an integer not divisible by $p$ (where $p$ is our fixed prime number). We have

$$\hat{G} = \begin{cases} \{r^{jp^t}, sr^k | 0 \leq j < l, 1 \leq k \leq 2n \}, & \text{if } p \neq 2; \\ \{r^{jp^t} | 0 \leq j < l \}, & \text{if } p = 2. \end{cases}$$

Thus, the $p$-regular classes of $G$ are

$$\begin{cases} \{r^{jp^t}, sr^{(l-j)p^t} \}; 0 \leq j \leq \frac{l}{2}, & \{sr^{2k} | 1 \leq k \leq n \}, \{sr^{2k+1} | 0 \leq k \leq n - 1 \}, \text{if } p \neq 2; \\ \{r^{jp^t}, sr^{(l-j)p^t} \}; 0 \leq j \leq \frac{l}{2}, & \text{if } p = 2. \end{cases}$$

For each $j$ and $h$, denote

$$\hat{\psi}_j = \psi_j|_{\hat{G}}, \quad \hat{\chi}_h = \chi_h|_{\hat{G}} \quad \text{and} \quad \hat{\psi}'_j = \psi'_j|_{\hat{G}}, \quad \hat{\chi}'_h = \chi'_h|_{\hat{G}},$$

and define $\epsilon = \begin{cases} 4, & \text{if } p \neq 2; \\ 1, & \text{if } p = 2. \end{cases}$

**Proposition 4.1.** The complete list of irreducible Brauer characters of $T_{4n}$ for even $n$ is

$$\hat{\chi}_h \quad (0 \leq h < \epsilon), \quad \hat{\psi}_j \quad (1 \leq j < \frac{l}{2}),$$
Proof. We first note that the restriction of a character of $T_{4n}$ to $\hat{T}_{4n}$ is a Brauer character and the number of all the irreducible Brauer characters is the number of $p$-regular classes of $T_{4n}$. Also, since $T_{4n}$ is solvable, by the Fong-Swan theorem, any irreducible Brauer character of $T_{4n}$ is the restriction of an ordinary irreducible character of $T_{4n}$.

The linear characters $\hat{\chi}_h$'s and $\hat{\chi}'_h$'s are obviously irreducible and distinct, by the character tables above. For characters, $\hat{\psi}_j$ and $\hat{\psi}'_j$, of dimension two we claim that they are all distinct and irreducible for all $1 \leq j < \frac{l}{2}$. By the character tables above, there is no need to separate the proof into the case of odd $n$, even $n$ or $p = 2, p \neq 2$, since $\hat{\psi}_j$ and $\hat{\psi}'_j$ are agree on the columns $r^k$'s and their values are zero outside these columns.

For the irreducibility issue, we suppose for a contradiction that

$$\hat{\psi}_j = \hat{\chi}_h + \hat{\chi}_k,$$

for some $0 \leq h, k < \epsilon$ and $1 \leq j < \frac{l}{2}$. Since $1 \leq j < \frac{l}{2}$, $l > 2$ and $r^{2p^i} \in \hat{T}_{4n}$. So, we can evaluate both sides of the above equation at $r^{2p^i}$ and obtain that

$$2 \cos \left( \frac{2p^i j \pi}{n} \right) = 2,$$

which is impossible because $\cos \left( \frac{2p^i j \pi}{n} \right) < 1$ for all $1 \leq j < \frac{l}{2}$.

Analogously, for the issue of distinction, we suppose for a contradiction that $\hat{\psi}_j = \hat{\psi}_i$, for some $1 \leq i < j < \frac{l}{2}$. We now evaluate both sides by $r^{p^i}$, which yields that

$$\cos \left( \frac{p^i j \pi}{n} \right) = \cos \left( \frac{p^i i \pi}{n} \right).$$

It implies that, for $\frac{p^i j \pi}{n}$ and $\frac{p^i i \pi}{n}$, their difference or their sum must be a multiple of $2\pi$. However, this is not the case because $1 \leq i < j < \frac{l}{2}$. $\Box$

**Theorem 4.2.** Let $G = T_{4n}$, $0 \leq h < \epsilon$ where $\epsilon = 4$ if $p \neq 2$ and $\epsilon = 1$ if $p = 2$, and put $\phi = \hat{\chi}_h$ or $\hat{\chi}'_h$. The space $V_\phi(G)$ has an o-basis if and only if at least one of the following holds:

(i) $\dim V = 1,$
Orthogonal Bases of Brauer Symmetry Classes of Tensors for Groups Having Cyclic Support

(ii) $p = 2$,

(iii) $2n$ is not divisible by $p$.

Proof. (i) If $\dim V = 1$, then $V_\phi(G) = \langle e_\alpha^\phi \mid \alpha \in \Gamma_{<1}^n \rangle$ has only at most one generator, namely, $e_\alpha^\phi$ where $\alpha = (1, 1, 1, \ldots, 1)$. So, $\dim V_\phi(G) \leq 1$, and thus, $V_\phi(G)$ has an o-basis.

(ii) If $p = 2$ then $\hat{G} = \langle r^p \rangle$. Since $\hat{G}$ is a subgroup of $G$ and $\phi$ is constant on $\hat{G}$, it follows by Proposition 2.7 that $V_\phi(G)$ has an o-basis.

(iii) Assume $p \neq 2$ and $2n$ is not divisible by $p$. Then $\hat{G} = G$ and consequently, these characters will be ordinary linear characters. Thus, $V_\phi(G)$ has an o-basis.

Conversely, we assume that $\dim V > 1$ and $p \neq 2$ and $2n$ is divisible by $p$. So, $r \notin \hat{G}$ and $\hat{G} = \{r^j, sr^k \mid 0 \leq j < l, 1 \leq k \leq 2n\} = \hat{G}^1$. We will show that $V_\phi(G)$ does not have an o-basis. For $\alpha = (1, 2, \ldots, 2, 2) \in \Gamma_{\text{dim } V}^n$, we have $G_\alpha = \{e\}$. Now, we concentrate on $\langle e_{\alpha\sigma}^\phi, e_\alpha^\phi \rangle$, for each $\sigma \in G$. We observe that $A = \{\mu \in \hat{G} \mid e \in \sigma \mu^{-1}\hat{G}\} = \{\mu \in \hat{G} \mid \sigma \in G_\mu\}$. Since $r^i = (sr^j)(sr^k) \in \hat{G}^2$ for each $0 \leq i < 2n$, $G \subseteq \hat{G}^2$ and hence $A \neq \emptyset$. Thus, by Corollary 2.3, we have

$$\langle e_{\alpha\sigma}^\phi, e_\alpha^\phi \rangle \neq 0 \text{ for each } \sigma \in G. \quad (4.1)$$

Next, we claim that $\{e_{\alpha\sigma}^\phi, e_\alpha^\phi \} \subseteq V_\phi(G)$ is a linearly independent set. We can set $e_\alpha^\phi = \sum_{r \in \hat{G}} e_{\alpha r}^\phi$ and $e_\alpha = \sum_{r \in \hat{G}} d_r e_\delta$ as $\{e_\delta \mid \delta \in \Gamma_{\text{dim } V}^n\}$ forms a basis for $V^{\otimes 4n}$. Since $\hat{G}^{-1} = \hat{G}$,

$$e_\alpha^\phi = \frac{\phi(1)}{|\hat{G}|} \sum_{\sigma \in \hat{G}} \phi(\sigma^{-1})e_{\alpha\sigma}.$$

Since $G_\alpha = \{e\}$, the elements $\alpha\sigma$ with $\sigma \in G$ are distinct. Also, since $r \notin \hat{G}$, $\alpha\sigma \neq \alpha r$ for all $\sigma \in \hat{G}$, which yields that $c_r = 0$. On the other hand, $G_{\alpha r} = r^{-1}G_\alpha r = \{e\}$, so for $r \in \hat{G}$, $(\alpha r)\sigma = \alpha r$ if and only if $\sigma = e$. This implies that $d_r = \frac{1}{|\hat{G}|} \neq c_r = 0$, which implies that $\{e_{\alpha\sigma}^\phi, e_\alpha^\phi \} \subseteq V_\phi(G)$ is a linearly independent set. Hence, $\dim V_\phi(G) \geq 2$.

By Proposition 2.5, if $V_\phi(G)$ has an o-basis, then it has an o-basis containing $e_\alpha^\phi$, but, by (4.1), this is not the case. So, $V_\phi(G)$ does not have an o-basis, and by Theorem 2.1, we complete the proof. 

For higher dimensional irreducible Brauer characters $\phi : \hat{G} \to \mathbb{C}$, we see that if $\dim V = 1$, then $V_\phi(G) = \langle e_\alpha^\phi \mid \alpha \in \Gamma_{<1}^n \rangle$ has only at most one generator, namely, $e_\alpha^\phi$ where $\alpha = (1, 1, 1, \ldots, 1)$. So, $\dim V_\phi(G) \leq 1$, and thus, $V_\phi(G)$ has an o-basis.

If $\dim V > 1$, we investigate a necessary condition of the existence of an o-basis for
dicyclic groups as follows.

**Proposition 4.3.** For $G = T_{2(l^p)}$ with $C \cap G_\gamma = \langle r^\gamma p^t \rangle$ where $t_\gamma = \frac{l}{|C \cap G_\gamma|}$, $\gamma \in \Delta$ and $\phi = \bar{\psi_b}, \bar{\psi'}_b$, where $1 \leq b < \frac{l}{2}$, we have that

$$\dim(V_{\gamma}^\phi(C)) = \begin{cases} 2, & \text{if } \frac{bt_\gamma}{l} \in \mathbb{Z}; \\ 0, & \text{if } \frac{bt_\gamma}{l} \notin \mathbb{Z}. \end{cases}$$

**Proof.** Since $G$ has cyclic support with $C = \langle r^p \rangle$ and $C \cap G_\gamma = \langle r^\gamma p^t \rangle$, for each $\phi$, we can compute the dimension by using Proposition 3.4. By characteristics and basic trigonometry identities, we compute that

$$v_j = \sum_{h \in C \cap G_\gamma} \phi(h x_\gamma^{-j}) = \sum_{m=1}^{t_\gamma} \phi(r^{(mt_\gamma - j)b_\gamma}) = \sum_{m=1}^{t_\gamma} 2 \cos \left( m \left( \frac{2b_\gamma}{t_\gamma} \right) \pi - \left( \frac{2b_\gamma}{t_\gamma} \right) \pi \right) = 2 \left( \frac{b_\gamma}{t_\gamma} \right) \cos \left( \left( \frac{2b_\gamma}{t_\gamma} \right) \pi \right), \quad \frac{bt_\gamma}{l} \in \mathbb{Z};$$

$$0, \quad \frac{bt_\gamma}{l} \notin \mathbb{Z}.$$.

So, if $\frac{bt_\gamma}{l} \notin \mathbb{Z}$, then $d_\gamma = t_\gamma$ and thus $\dim(V_{\gamma}^\phi(C)) = t_\gamma - t_\gamma = 0$.

For $d_\gamma$ in which $\frac{bt_\gamma}{l} \in \mathbb{Z}$, we have that $\sum_{j=0}^{t_\gamma-1} v_j e^{\frac{2\pi i j}{l t_\gamma}} = 0$ if and only if

$$\sum_{j=0}^{t_\gamma-1} 2 \cos \left( \left( \frac{2b_\gamma}{l} \right) \pi \right) \cos \left( \left( \frac{2s_j}{t_\gamma} \right) \pi \right) \text{ and } \sum_{j=0}^{t_\gamma-1} 2 \cos \left( \left( \frac{2b_\gamma}{l} \right) \pi \right) \sin \left( \left( \frac{2s_j}{t_\gamma} \right) \pi \right),$$

are simultaneously zero. Since $\frac{bt_\gamma}{l} \in \mathbb{Z}$, the second sum is always zero and the first sum is zero for all $0 \leq s < t_\gamma$, except for $\frac{b_\gamma}{l} + \frac{s}{t_\gamma} \notin \mathbb{Z}$; (i.e., except for $s = t_\gamma - \frac{bt_\gamma}{l}$ or $s = \frac{bt_\gamma}{l}$), because $0 < \frac{b_\gamma}{l} + \frac{s}{t_\gamma} < 2$ and $-1 < \frac{b_\gamma}{l} - \frac{s}{t_\gamma} < 1$. Hence, $d_\gamma = t_\gamma - 2$, and thus, the results follow. $\square$

There is no surprise with the assertion that $\dim(V_{\gamma}^\phi(C)) = 0$ for which $\frac{bt_\gamma}{l} \notin \mathbb{Z}$ because:

**Proposition 4.4.** For $G = T_{2(l^p)}$ with $C = \langle r^p \rangle$ and $C \cap G_\gamma = \langle r^\gamma p^t \rangle$ where $t_\gamma = \frac{l}{|C \cap G_\gamma|}$, $\gamma \in \Delta$ and $\phi = \bar{\psi}_b, \bar{\psi'}_b$, where $1 \leq b < \frac{l}{4}$, we have that, for each $\sigma \in C$,

$$\sigma^\phi_{\gamma} = 0 \text{ if and only if } \frac{bt_\gamma}{l} \notin \mathbb{Z}.$$
Orthogonal Bases of Brauer Symmetry Classes of Tensors for Groups Having Cyclic Support

Proof. Let \( \sigma \in C \) and \( \gamma \in \Delta \). By (3.1),

\[
\langle e^\phi_{\gamma | \sigma}, e^\phi_{\gamma | \sigma} \rangle = \frac{|\sigma(\phi)|^2}{|G|^2} \sum_{\mu \in C} \sum_{r \in C \cap G, \gamma} \phi(\mu) \bar{\phi}(\mu r) \\
= 4 \left( \frac{|\sigma(\phi)|^2}{|G|^2} \sum_{l=1}^{b} \cos \left( \frac{2l\pi}{b} \right) \cos \left( \frac{2k(b+2l)}{b} \pi \right) \right) \\
= \left\{ \begin{array}{ll}
0, & \text{if } \frac{b}{l} \in \mathbb{Z}, \\
(\frac{4}{b}) \left( \frac{|\sigma(\phi)|^2}{|G|^2} \sum_{j=1}^{l} \cos^2 \left( \frac{2l\pi}{b} \right) \right), & \text{if } \frac{b}{l} \notin \mathbb{Z},
\end{array} \right.
\]

which completes the proof. \( \square \)

Now, by the above propositions, we achieve the main conclusion.

**Theorem 4.5.** Let \( G = T_{2n} \), where \( 2n = lp^t \) with \( t \) an integer not divisible by \( p \) and let \( \phi = \psi_b, \psi'_b \), where \( 1 \leq b < \frac{t}{2} \). Then, \( V^\phi(G) \) has an o-basis if and only if \( \nu_2(\frac{b}{t}) < 0 \).

**Proof.** By Proposition 4.2, it is enough to focus on \( V^\phi_\gamma(C) \). Also, in the proof of Proposition 4.3, we have \( V^\phi_\gamma(C) = \langle e^\phi_{\gamma | \sigma} | j = 1, 2, \ldots, t_\gamma \rangle \), where \( \sigma_j = r^j p^t \) and \( t_\gamma = \left\lfloor \frac{G}{C \cap G \sigma_j} \right\rfloor \). Again, by (3.1) and the character tables, we compute that, for \( 1 \leq i, j \leq t_\gamma \),

\[
\langle e^\phi_{\gamma | \sigma_i}, e^\phi_{\gamma | \sigma_j} \rangle = 0 \iff \sum_{g \in C} \phi(g \sigma_i) \bar{\phi}(g \sigma_j) = 0 \\
\iff \sum_{k=0}^{l-1} \phi(r^{(k+1)p^t} \sigma_j) \bar{\phi}(r^{(k+1)p^t} \sigma_i) = 0 \\
\iff \sum_{k=0}^{l-1} 2 \cos \left( (k + j) \frac{2b \pi}{bt} \right) \cos \left( (k + j) \frac{2b \pi}{bt} \right) = 0 \\
\iff l \cos \left( (i - j) \frac{2b \pi}{bt} \right) = 0.
\]

By Proposition 4.3, \( \dim(V^\phi_\gamma(C)) = 2 \) for each \( \gamma \) such that \( \frac{bt}{l} \in \mathbb{Z} \). So, if \( V^\phi_\gamma(G) \) contains an o-basis, then there exist \( \gamma \) and distinct \( 1 \leq i, j \leq t_\gamma \) such that \( \cos \left( (i - j) \frac{2b \pi}{bt} \right) = 0 \), which clearly implies that \( \nu_2(\frac{b}{t}) < 0 \). On the other hand, suppose \( \nu_2(\frac{b}{t}) = -k \), for some \( k \in \mathbb{N} \). Then \( \frac{b}{l} = \frac{m}{2k \pi} \) for some odd integer \( m \). Since the existence of an o-basis depends on \( \gamma \) for which \( \frac{bt}{l} \in \mathbb{Z} \), \( 2^{k+1} \) is always a divisor of \( t_\gamma \). Thus, we can choose \( i_0 = 2^{k-1} + 1 \) and \( j_0 = 1 \) so that \( \cos \left( (i_0 - j_0) \frac{2b \pi}{bt} \right) = 0 \). By Proposition 4.4, \( e^\phi_{\gamma | i_0} \) and \( e^\phi_{\gamma | j_0} \) are non zero and hence, by the above fact, \( \langle e^\phi_{\gamma | (2^{k-1}+1)p^t}, e^\phi_{\gamma | p^t} \rangle \) forms an o-basis for \( V^\phi_\gamma(C) \). \( \square \)

5. Dihedral group \( D_m \). We first collect some facts about the Brauer characters of the dihedral groups \( D_{2m} \) from [10]. We follow the notions of [10] in this section. A presentation of the dihedral groups \( D_m \) having order \( 2m \), is given by \( D_m = \langle r, s | r^m = s^2 = 1, sr = r^{-1} s \rangle \). The ordinary character table of \( D_m \) is:
Table III: The character table of $D_m$.

<table>
<thead>
<tr>
<th>Characters</th>
<th>$r^k$</th>
<th>$sr^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_0$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_2$</td>
<td>$(-1)^k$</td>
<td>$(-1)^k$</td>
</tr>
<tr>
<td>$\psi_2$</td>
<td>$(-1)^k$</td>
<td>$(-1)^{k+1}$</td>
</tr>
</tbody>
</table>

$\chi_h = 2 \cos \frac{2\pi kh}{m}$

$(1 \leq h < \frac{m}{2})$

We write $m = lp^t$, where $l$ is not divisible by prime number $p$, as before. The set of all $p$-regular elements of $D_m$ are

$$\overline{D_m} = \{ \{ r^{jp}, sr^k \mid 0 \leq j < l, 0 \leq k < \frac{m}{2} \}, \quad p \neq 2; \}
\{ \{ r^{jp} \mid 0 \leq j < l \}, \quad p = 2. \}

The complete list of irreducible Brauer characters of $D_m$ is, [10],

$$\hat{\psi}_j (0 \leq j < \epsilon), \hat{\chi}_h (1 \leq h < \frac{l}{2}),$$

where $\hat{\psi}_j = \psi_j |_{\overline{D_m}}$, $\hat{\chi}_h = \chi_h |_{\overline{D_m}}$ and

$$\epsilon = \begin{cases} 4, & l \text{ even, } p \neq 2; \\ 2, & l \text{ odd, } p \neq 2; \\ 1, & p = 2. \end{cases}$$

Necessary and sufficient condition for the existence of an $o$-basis for Brauer characters of dimension one is provided in [10]. Precisely, for $\phi = \hat{\psi}_j, \hat{\chi}_h$, where $0 \leq j < \epsilon$, the space $V_\phi(D_m)$ has an $o$-basis if and only if $\dim V = 1$ or $p = 2$ or $m$ is not divisible by $p$.

Necessary and sufficient condition for the existence of an $o$-basis for Brauer character of dimension two for $D_m$ can be found in [10]. But it can also be obtained by very similar method applied on $T_{4n}$ as we presented in §4. This is because $\phi$ has a cyclic support for each $\phi = \hat{\chi}_h$, where $0 \leq h < \frac{l}{2}$, and all values in the character tables of both groups are consistent on $C = < r^k >$. Thus, by changing $m$ to $2n$ and $h$ to $b$, each step of the computation for dimensions of $V_\phi^o(D_m)$ and the condition for the existence becomes the same. This yields

**Theorem 5.1.** Let $G = D_m$, where $m = lp^t$ with $l$ an integer not divisible by $p$ and let $\phi = \hat{\chi}_h$, where $1 \leq h < \frac{l}{2}$. Then, $V_\phi(G)$ has an $o$-basis if and only if $\nu_2(\frac{b}{h}) < 0$. Also, for each $\sigma \in C$, $e_\phi^o \neq 0$ if and only if $\frac{b}{h} \in \mathbb{Z}$. 
Orthogonal Bases of Brauer Symmetry Classes of Tensors for Groups Having Cyclic Support

6. Irreducible Brauer character of $SD_{8n}$. A presentation for $SD_{8n}$ for $n \geq 2$ is given by $SD_{8n} = \langle a, b \mid a^{4n} = b^2 = e, bab = a^{2n-1} \rangle$. All $8n$ elements of $SD_{8n}$ may be given by

$$SD_{8n} = \{ e, a, a^2, \ldots, a^{4n-1}, b, ba, ba^2, \ldots, ba^{4n-1} \}.$$ 

The embedding of $SD_{8n}$ into the symmetric group $S_{8n}$ is given by $T(a)(t) := t+1$ and $T(b)(t) := (2n-1)t$, where $\oplus$ is the remainder of $m$ divided by $4n$. We write $4n = lp^i$ with prime $p$ and integer $l$ not divisible by $p$ and denote by $\tilde{SD}_{8n}$ the set of all $p$-regular elements of $SD_{8n}$. It is not hard to see that $\tilde{SD}_{8n}$ is given by

$$\tilde{SD}_{8n} = \left\{ \begin{array}{ll}
\{a^{lp^i}b^k \mid 0 \leq j < l; 0 \leq k < 4n\}, & \text{if } p \neq 2; \\
\{a^{lp^i} \mid 0 \leq j < l\}, & \text{if } p = 2.
\end{array} \right.$$ 

By direct calculation, we have the following property.

**Proposition 6.1.** The $p$-regular classes of $SD_{8n}$, $n \geq 2$ and $4n = lp^i$, are as follows:

**Case 1:** $p$ is odd prime.

- If $n$ is even (i.e., $\frac{1}{2} \in \mathbb{Z}$), then there are $\frac{n}{2} + 3$ $p$-regular classes. Precisely,
  - 2 classes of size one being $\{e\}$ and $\{a^{2p^i}\}$,
  - $\frac{n}{2} - 1$ classes of size two being $[a^{lp^i}] = \{a^{lp^i}, a^{l(j-2)p^i}\}; j \in \{2, 4, 6, \ldots, \frac{n}{2} - 2\}$,
  - $\frac{n}{4}$ classes of size two being $[a^{3p^i}] = \{a^{lp^i}, a^{4(j-1)p^i}\}; j \in \{1, 3, 5, \ldots, \frac{n}{2} - 1\}$,
  - $\frac{n}{4}$ classes of size two being $[a^{5p^i}] = \{a^{lp^i}, a^{4(j-2)p^i}\}; j \in \{\frac{n}{2} + 1, \frac{n}{2} + 3, \ldots, \frac{n}{2} - 1\}$ and
  - 2 classes of size 2n being $[b] = \{ba^{2i} \mid i = 0, 1, 2, \ldots, 2n - 1\}$ and $[ba] = \{ba^{2i+1} \mid i = 0, 1, 2, \ldots, 2n - 1\}$.

- If $n$ is odd (i.e., $\frac{1}{2}$ is odd), then there are $\frac{n}{2} + 6$ $p$-regular classes. Precisely,
  - 4 classes of size one being $\{e\}$, $\{a^{2p^i}\}$, $\{a^{3p^i}\}$ and $\{a^{4p^i}\}$,
  - $\frac{n}{4} - 1$ classes of size two being $[a^{5p^i}] = \{a^{lp^i}, a^{4(j-2)p^i}\}; j \in \{2, 4, 6, \ldots, \frac{n}{2} - 2\}$,
  - $\frac{n}{4}$ classes of size two being $[a^{3p^i}] = \{a^{lp^i}, a^{4(j-1)p^i}\}; j \in \{1, 3, 5, \ldots, \frac{n}{2} - 1\}$,
  - $\frac{n}{4}$ classes of size two being $[a^{5p^i}] = \{a^{lp^i}, a^{4(j-2)p^i}\}; j \in \{\frac{n}{2} + 1, \frac{n}{2} + 3, \ldots, \frac{n}{2} - 2\}$ and
  - 4 classes of size $n$ being $[b] = \{ba^{4i} \mid i = 0, 1, 2, \ldots, n - 1\}$, $[ba] = \{ba^{4i+1} \mid i = 0, 1, 2, \ldots, n - 1\}$, $[ba^2] = \{ba^{4i+2} \mid i = 0, 1, 2, \ldots, n - 1\}$ and $[ba^3] = \{ba^{4i+3} \mid i = 0, 1, 2, \ldots, n - 1\}$.

**Case 2:** $p = 2$. There are $\frac{n+2}{2} p$-regular classes. Precisely, there is 1 class of size one, $\{e\}$, and there are $\frac{n+2}{4}$ classes of size two, $\{a^{lp^i}, a^{4(j-1)p^i}\}; 1 \leq j \leq \frac{n+2}{4}$. 
The ordinary irreducible character of $SD_{8n}$ are given by (see [11]):

**Table IV:** The character table for $SD_{8n}$, when $n$ is even.

<table>
<thead>
<tr>
<th>Conjugacy classes, Characters</th>
<th>$[a^r]; \ r \in C_1$</th>
<th>$[a^r]; \ r \in C_{odd}^1$</th>
<th>$[b]$</th>
<th>$[ba]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_h$, where $h \in C_{even}$</td>
<td>$2 \cos \left( \frac{h \pi}{2n} \right)$</td>
<td>$2 \cos \left( \frac{h \pi}{2n} \right)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_h$, where $h \in C_{odd}$</td>
<td>$2 \cos \left( \frac{h \pi}{2n} \right)$</td>
<td>$2i \sin \left( \frac{h \pi}{2n} \right)$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table V:** The character table for $SD_{8n}$, when $n$ is odd.

<table>
<thead>
<tr>
<th>Conjugacy classes, Characters</th>
<th>$[a^r]; \ r \in C_1'$</th>
<th>$[a^r]; \ r \in C_{odd}^{1'}$</th>
<th>$[b]$</th>
<th>$[ba]^2$</th>
<th>$[ba]^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0'$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1'$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2'$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3'$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_4'$</td>
<td>$(-1)^{\frac{r}{2}}$</td>
<td>$i^r$</td>
<td>1</td>
<td>$-i$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5'$</td>
<td>$(-1)^{\frac{r}{2}}$</td>
<td>$i^r$</td>
<td>-1</td>
<td>$i$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_6'$</td>
<td>$(-1)^{\frac{r}{2}}$</td>
<td>$(-i)^r$</td>
<td>1</td>
<td>$-i$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_7'$</td>
<td>$(-1)^{\frac{r}{2}}$</td>
<td>$(-i)^r$</td>
<td>-1</td>
<td>i</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_h'$, where $h \in C_{even}$</td>
<td>$2 \cos \left( \frac{h \pi}{2n} \right)$</td>
<td>$2 \cos \left( \frac{h \pi}{2n} \right)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_h'$, where $h \in C_{odd}$</td>
<td>$2 \cos \left( \frac{h \pi}{2n} \right)$</td>
<td>$2i \sin \left( \frac{h \pi}{2n} \right)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where $C_1 = \{0, 2, 4, \ldots, 2n\}$, $C_{even}^1 := C_1 \setminus \{0, 2n\}$, $C_{odd}^{1'} = \{1, 3, 5, \ldots, n, 2n+1, 2n+3, 2n+5, \ldots, 3n\}$, $C_{odd}^{1} = \{1, 3, 5, \ldots, n-1, 2n+1, 2n+3, 2n+5, \ldots, 3n-1\}$.

For each $k$ and $h$, put $\hat{\chi}_k = \chi_k |_{SD_{8n}}$, $\hat{\chi}'_k = \chi'_k |_{SD_{8n}}$ and $\hat{\psi}_k = \psi_k |_{SD_{8n}}$, $\hat{\psi}'_k = \psi'_k |_{SD_{8n}}$. Moreover, for odd prime $p$ and $4n = lp^t$ such that $l$ is not divisible
Orthogonal Bases of Brauer Symmetry Classes of Tensors for Groups Having Cyclic Support

by p, we define \( E := \{2, 4, 6, \ldots, \frac{l}{2} - 2\} \), \( O_1^2 := \{1, 3, 5, \ldots, \frac{l}{2} - \epsilon\} \), \( O_2^2 := \{\frac{l}{2} + 1, \frac{l}{2} + 3, \ldots, \frac{l}{2} + \frac{l}{4} - \epsilon\} \), where \( \epsilon = 1 \) if \( n \) is even and \( \epsilon = 2 \) if \( n \) is odd.

**Proposition 6.2.** Let \( \text{IBr}(SD_{8n}) \) be the set of all distinct irreducible Brauer characters of \( SD_{8n} \). Then,

\[
\text{IBr}(SD_{8n}) = \begin{cases} 
\{ \hat{\chi}_k, \hat{\psi}_{j,p^t} | 0 \leq k \leq 3, j \in E \cup O_1^2 \cup O_2^2 \}, & \text{if } p \neq 2 \text{ and } n \text{ is even}; \\
\{ \hat{\chi}_k, \hat{\psi}_{j,p^t} | 0 \leq k \leq 7, j \in E \cup O_1^2 \cup O_2^2 \}, & \text{if } p \neq 2 \text{ and } n \text{ is odd}; \\
\{ \hat{\chi}_k, \hat{\psi}_{j,p^t} | 0 \leq j \leq \frac{l}{2} \}, & \text{if } p = 2 \text{ and } n \text{ is even}; \\
\{ \hat{\chi}_k, \hat{\psi}_{j,p^t} | 0 \leq j \leq \frac{l-1}{2} \}, & \text{if } p = 2 \text{ and } n \text{ is odd}. 
\end{cases}
\]

**Proof.** We first note that the restriction of a character of \( SD_{8n} \) to \( SD_{8n} \) is a Brauer character and the order of the set \( \text{IBr}(SD_{8n}) \) is the number of \( p \)-regular classes of \( SD_{8n} \). Also, since \( SD_{8n} \) is solvable, by Fong-Swan theorem, any element in \( \text{IBr}(SD_{8n}) \) is the restriction of an ordinary irreducible character of \( SD_{8n} \).

Each \( \hat{\chi}_k \)'s and \( \hat{\psi}_k \)'s are obviously irreducible and clearly distinct, by the character tables above. For characters of dimension two, \( \hat{\psi}_{j,p^t} \) where \( p \) is an odd prime and \( n \) is even, we claim that those are irreducible. We suppose for a contradiction that \( \hat{\psi}_{j,p^t} = \hat{\chi}_i + \hat{\chi}_k \) for some \( j \in E \cup O_1^2 \cup O_2^2 \) and \( 0 \leq i, k \leq 3 \). Evaluating both sides at \( a^{2p^t} \) yields that

\[
2 \cos \frac{jp^t \cdot 2p^t \pi}{2n} = 2.
\]

That is \( \cos \frac{4jp^t \pi}{2n} = 1 \), so \( 2j \) is a multiple of \( l \). However, since \( 2j < l \) for \( j \in E \cup O_1^2 \) and \( l < 2j < 2l \) for \( j \in O_2^2 \), this is a contradiction. We use similar arguments to show that all the remaining cases, \( \hat{\psi}_{j,p^t} \)'s are irreducible.

Next, we aim to show that all elements in \( \text{IBr}(SD_{8n}) \) shown in the proposition are distinct. For the case odd prime \( p \) and even \( n \), we suppose that \( \hat{\psi}_{j,p^t} = \hat{\psi}_{i,p^t} \) for some \( i, j \in E \cup O_1^2 \cup O_2^2 \). It is clear (by the character table) that \( i, j \) either both are even or both are odd. If \( i, j \) are even, we evaluate both sides at \( a^{p^t} \) and then we get

\[
\sin \frac{p^t(i + j)\pi}{l} \sin \frac{p^t(j - i)\pi}{l} = 0.
\]

Since \( \gcd(l, p^t) = 1 \) and \( \frac{i + j}{l} \) and \( \frac{j - i}{l} \) can not be positive integers for each \( i, j \in E \), \( i = j \). If \( i, j \) are odd, we evaluate both sides at \( a^{p^t} \), and then we get

\[
\sin \frac{p^t(j - i)\pi}{l} \cos \frac{p^t(j + i)\pi}{l} = 0.
\]

Since \( \gcd(l, p^t) = 1 \) and \( \frac{i + j}{2} \neq \frac{l}{2}, \frac{3l}{2} \) for \( i, j \in O_1^2 \cup O_2^2 \) and \( \frac{j - i}{2} \) can not be positive integer, \( i = j \). Again, similar arguments work for all the remaining cases. \( \square \)
7. Existence of an o-basis for the class of tensors using a Brauer character of the $SD_{8n}$. In the following theorem, we denote

$$
\epsilon = \begin{cases} 
3 & \text{if } p \neq 2, \ n \text{ even} \\
7 & \text{if } p \neq 2, \ n \text{ odd} \\
1 & \text{if } p = 2.
\end{cases}
$$

**Theorem 7.1.** Let $\dim V > 2$, $G = SD_{8n}$, $0 \leq j \leq \epsilon$, and put $\phi = \hat{\chi}_j$ or $\hat{\chi}'_j$. Then, space $V_\phi(G)$ has an o-basis if and only if $p = 2$ or $4n$ is not divisible by $p$.

**Proof.** If $p = 2$ then $\hat{G} = \langle \sigma^p \rangle$. Since $\hat{G}$ is a subgroup of $G$ and $\phi$ is constant on $\hat{G}$, by Proposition 2.1 $V_\phi(G)$ has an o-basis. Assume $p \neq 2$ and $4n$ is not divisible by $p$. Then $G = \hat{G}$ and consequently, these characters will be ordinary linear characters. Thus, $V_\phi(G)$ has an o-basis.

Conversely, suppose that $p \neq 2$ and $4n$ is divisible by $p$. We aim to show that $V_\phi(G)$ does not have an o-basis by showing that there exists $\alpha \in \Gamma^4_k$ such that $V_\phi^\alpha(G)$ does not have an o-basis and then apply Theorem 2.1 to conclude the results.

Let $\alpha = (1, 2, 2, \ldots, 2, 3)$. Since $\dim V > 2$ and $4n \geq 4$, $\alpha \in \Gamma^{4n}_{\dim V}$. We also choose a representative $\Delta$ so that $\alpha \in \Delta$. We observe that to fix $\alpha$, each $\sigma \in G$ must fix the first and the last position of $\alpha$. It is clear that element of the form $a^k$ satisfying the condition is only $\epsilon$. For elements of the form $ba^k$, they must satisfy $T(ba^k)(1) = 1$ and $T(ba^k)(4n) = 4n$. By using $T(ba^k)(t) = (2n-1)(k+t)$, we conclude that $G_\alpha = \{e\}$. Since $\phi$ is a restriction of a linear character and $G_\alpha = \{e\}$, by Corollary 2.3 to show that $(e_\phi^\alpha, e_\alpha^\phi) \neq 0$ for each $\sigma \in G$, it suffices to show that $A = \{\mu \in \hat{G} | e \in \sigma^{-1} \hat{G}\} \neq \emptyset$. This is simple because $p \neq 2$ and $4n$ is divisible by $p$, so $\hat{G} = \{a^k | 0 \leq k < m\} = \hat{G}^{-1}$ and then $A = \{\mu \in \hat{G} | e \in \hat{G} \mu\}$. Since $e \in \sigma^{-1} \hat{G}$ if and only if $\sigma \in \hat{G}^{-1} \mu = \hat{G} \mu$ and thus for arbitrary $0 \leq k < 4n$, we have $a^k = ba^kba^k \in \hat{G}^2$ and $ab^k = a^oba^k \in \hat{G}^2$, so $G \subseteq \hat{G}^2$. That is $A \neq \emptyset$. So,

$$
(e_\phi^\alpha, e_\alpha^\phi) \neq 0 \quad \text{for each } \sigma \in G. \quad (7.1)
$$

Next, to show that $\{e_\phi^\alpha, e_\alpha^\phi\} \subseteq V_\phi^\alpha(G)$ is a linearly independent set, we set $e_\phi^\alpha = \sum_{\delta} e_\phi^{\alpha \delta}$ and $e_\alpha^\phi = \sum_{\delta} d_\delta e_\delta$, as $\{e_\delta | \delta \in \Gamma^m_k\}$ forms a basis for $V^\otimes m$. Since $G^{-1} = \hat{G}$,

$$
eulsion{e_\phi^\alpha} = \frac{\phi(1)}{|\hat{G}|} \sum_{\sigma \in \hat{G}} \phi(\sigma^{-1}) e_{\alpha \sigma}.
$$

Since $G_\alpha = \{e\}$, the elements $\alpha \sigma$ with $\sigma \in G$ are distinct. Also, since $a \notin \hat{G}$, $\alpha \sigma \neq \alpha a$ for all $\sigma \in \hat{G}$, which yields that $c_\alpha = 0$. On the other hand, $G_{\alpha a} = a^{-1}G_\alpha a = \{e\}$,
Orthogonal Bases of Brauer Symmetry Classes of Tensors for Groups Having Cyclic Support

so for \( a \in \hat{G} \), \((\alpha a)\sigma = a\sigma\) if and only if \( \sigma = e \). This implies that \( d_a = \frac{1}{|\sigma|} \neq c_{aa} = 0 \). Thus, \( \{e^{\phi}_a, e^{\phi}_a\} \subseteq V^{\phi}_a(G) \) is a linearly independent set, and hence, \( \dim V^{\phi}_a(G) \geq 2 \).

By Proposition 2.4, if \( V^{\phi}_a(G) \) were to have an o-basis, then it would have an o-basis containing \( e^{\phi}_a \), but, by (7.1), this is not the case. So, \( V^{\phi}_a(G) \) does not have an o-basis, which completes the proof. \( \square \)

**Remark 7.2.** Theorem 7.1 shows that if \( \dim V > 2 \), unlike the case for an irreducible character, it is possible that \( V^{\phi}_a(G) \) has no o-basis when \( \phi \) is a linear Brauer character. This holds when \( \dim V = 2 \) as well. To observe this, we let \( p \neq 2 \) and \( 4n \) is divisible by \( p \) and \( \phi = \hat{\chi}_0 \) or \( \hat{\chi}_0' \). Consider \( \alpha = (1, 2, \ldots, 2, 2) \in I_{4n}^{\dim V} \). Thus, for such \( \alpha \), we have \( G_{\alpha} = \{1, a^{2n+2\beta}\} \). Now by similar calculations done in Theorem 7.1, we have \( \langle e^{\phi}_{\alpha}, e^{\phi}_{\alpha}\rangle \neq 0 \).

For the remaining of this section, we denote

\[
\Pi = \begin{cases} 
\{jp^t | j \in E \cup O_1 \cup O_2\}, & \text{if } p \neq 2 \text{ and } n \text{ even}; \\
\{jp^t | j \in E \cup O_2 \cup O_2\}, & \text{if } p \neq 2 \text{ and } n \text{ odd}; \\
\{jp^t | 0 \leq j \leq \left\lfloor \frac{p-1}{2} \right\rfloor\}, & \text{if } p = 2.
\end{cases}
\]

For \( V^{\phi}_a(SD_{8n}) \), where \( \phi = \hat{\psi}_h, \hat{\psi}_h' \) such that \( h \in \Pi \) is even, the condition for the existence of \( \Pi \) can be obtained in the same manner as \( T_{4n} \) and \( D_{m} \). This is because \( \phi \) has a cyclic support and all values in the character tables of those groups are consistence on \( C = \langle a^4 \rangle \) if \( h \) is even. Then, we have:

**Theorem 7.3.** Let \( G = SD_{4n} \), where \( 4n = lp^t \) with \( l \) an integer not divisible by \( p \) and let \( \phi = \hat{\psi}_h, \hat{\psi}_h' \) such that \( h \in \Pi \) be even. Then, \( V^{\phi}_a(G) \) has an o-basis if and only if \( \text{v}_2(\frac{2h}{p}) < 0 \). Also, for each \( \sigma \in C, e^{\phi}_{a\sigma} \neq 0 \) if and only if \( \frac{ht}{p} \in \mathbb{Z} \).

For the case where \( h \in \Pi \) is odd, we first compute the dimension of \( V^{\phi}_a(SD_{8n}) \).

**Proposition 7.4.** Let \( G = SD_{4n} \), where \( 4n = lp^t \) with \( p \nmid l \) and let \( \phi = \hat{\psi}_h, \hat{\psi}_h' \) such that \( h \in \Pi \) be odd. For \( \gamma \in \Delta \) such \( C \cap G_{\gamma} = \langle a^4 \rangle \), where \( t_{\gamma} = \frac{1}{4\gamma} \), then

\[
\dim(V^{\phi}_a(C)) = \begin{cases} 
4, & \text{if } \frac{ht_{\gamma}}{p} \in \mathbb{Z}; \\
0, & \text{if } \frac{ht_{\gamma}}{p} \notin \mathbb{Z}.
\end{cases}
\]

**Proof.** Since \( G \) has cyclic support with \( C = \langle a^4 \rangle \) and \( C \cap G_{\gamma} = \langle a^4 \rangle \), for each \( \phi \), we can compute the dimension by using Proposition 3.3. By character tables, we compute \( v_j = \sum_{h \in C \cap G_{\gamma}} \phi(ht_{\gamma}^{-1}) \), for the different case of \( j \) and \( t_{\gamma} \). If \( t_{\gamma} \) is odd,
then, for even $j$,

$$v_j = \sum_{m=1}^{l/t\gamma} \phi(r(m t\gamma - j)p')$$

$$= i \sum_{k=1}^{l/2t\gamma} 2 \sin \left( (2k - 1) \left( \frac{2ht\gamma}{l} \right) \pi - \left( \frac{2h}{l} \pi \right) \right)$$

$$+ \sum_{k=1}^{l/2t\gamma} 2 \cos \left( (2k)(\frac{2ht\gamma}{l}) \pi - \left( \frac{2h}{l} \pi \right) \right)$$

$$= 0,$$

and for odd $j$,

$$v_j = \sum_{m=1}^{l/t\gamma} \phi(r(m t\gamma - j)p')$$

$$= \sum_{k=1}^{l/2t\gamma} 2 \cos \left( (2k - 1) \left( \frac{2ht\gamma}{l} \right) \pi - \left( \frac{2h}{l} \pi \right) \right)$$

$$+ i \sum_{k=1}^{l/2t\gamma} 2 \sin \left( (2k)(\frac{2ht\gamma}{l}) \pi - \left( \frac{2h}{l} \pi \right) \right)$$

$$= 0,$$

since $\frac{2ht\gamma}{l} \not\in \mathbb{Z}$ (because $h, t\gamma$ are odd and then $4 \mid l$). So, if $t\gamma$ is odd (i.e., $\frac{ht\gamma}{l} \not\in \mathbb{Z}$), then $d_\gamma = t_\gamma$ and thus dim$(V_{t\gamma}^\phi(C)) = t_\gamma - t_\gamma = 0$.

Similarly, if $t_\gamma$ is even, we compute that

$$v_j = \begin{cases} 
0, & \frac{ht\gamma}{l} \not\in \mathbb{Z}; \\
-2i \frac{\sin \left( \frac{2ht\gamma}{l} \pi \right)}{\frac{ht\gamma}{l} \in \mathbb{Z}} 
\end{cases}$$

and $j$ odd.

$$\frac{2t}{t_\gamma} \in \mathbb{Z} \text{ and } j \text{ even.}$$

So, if $t_\gamma$ is even and $\frac{ht\gamma}{l} \not\in \mathbb{Z}$, then $d_\gamma = t_\gamma$ and thus dim$(V_{t\gamma}^\phi(C)) = t_\gamma - t_\gamma = 0$. For $d_\gamma$ in which $t_\gamma$ is even and $\frac{ht\gamma}{l} \in \mathbb{Z}$, we have that $\sum_{j=0}^{t_\gamma-1} v_j e^{2\pi i j / t_\gamma} = 0$ if and only if

$$\frac{1}{t_\gamma} \left[ \sum_{k=0}^{t_\gamma-1} 2 \cos \left( \left( \frac{4hk}{t_\gamma} \right) \pi \right) \cos \left( \left( \frac{4hk}{t_\gamma} \right) \pi \right) + \sum_{k=0}^{t_\gamma-1} 2 \sin \left( \left( \frac{2hk+1}{t_\gamma} \right) \pi \right) \sin \left( \left( \frac{2s(2k+1)}{t_\gamma} \right) \pi \right) \right],$$

and

$$\frac{1}{t_\gamma} \left[ \sum_{k=0}^{t_\gamma-1} 2 \cos \left( \left( \frac{4hk}{t_\gamma} \right) \pi \right) \sin \left( \left( \frac{4hk}{t_\gamma} \right) \pi \right) - \sum_{k=0}^{t_\gamma-1} 2 \sin \left( \left( \frac{2hk+1}{t_\gamma} \right) \pi \right) \cos \left( \left( \frac{2s(2k+1)}{t_\gamma} \right) \pi \right) \right],$$

are simultaneously zero. Since $\frac{ht\gamma}{l} \in \mathbb{Z}$, the second sum is always zero and the first sum is zero for each $0 \leq s < t_\gamma$ except for $2 \left( \frac{h}{t} + \frac{\gamma}{t_\gamma} \right) \in \mathbb{Z}$ or $2 \left( \frac{h}{t} - \frac{\gamma}{t_\gamma} \right) \in \mathbb{Z}$. Since $0 \leq \frac{\gamma}{t_\gamma} < 1$,

$$\frac{2\gamma}{t_\gamma} \leq 2 \left( \frac{h}{t} + \frac{\gamma}{t_\gamma} \right) < \frac{2\gamma}{t_\gamma} + 2 \text{ and } \frac{2\gamma}{t_\gamma} - 2 \left( \frac{h}{t} - \frac{\gamma}{t_\gamma} \right) \leq \frac{2\gamma}{t_\gamma}.$$
Precisely, if \( s \) belongs to 
\[
\left\{ s_1 := \frac{t_2}{2} \left\lceil \frac{2h}{l} \right\rceil - \frac{ht_2}{l}, s_2 := \frac{t_2}{2} \left\lceil \frac{2h}{l} \right\rceil - \frac{ht_2}{l} + \frac{t_2}{2}, s_3 := \frac{ht_2}{l} - \frac{t_2}{2} \left\lfloor \frac{2h}{l} \right\rfloor, s_4 := \frac{ht_2}{l} - \frac{t_2}{2} \left\lfloor \frac{2h}{l} \right\rfloor + \frac{t_2}{2} \right\},
\]
then the first sum will be not zero. Here, \([r] \) and \( \lfloor r \rfloor \) are the ceiling function and floor function of the real number \( r \), respectively. Since \( t_\gamma > 0 \) and \( \frac{2h}{l} \notin \mathbb{Z} \) for each odd \( h \in \mathbb{N} \), \( s_1, s_2, s_3, s_4 \) are all distinct. Hence, \( d_\gamma = t_\gamma - 4 \), and thus, the results follow. \( \square \)

Now, we have:

**Theorem 7.5.** Let \( G = SD_{8n}, \) where \( 4n = lp^k \) with \( p \nmid l \) and let \( \phi = \hat{\psi}_h, \hat{\psi}_h^* \), where \( h \in \mathbb{N} \) be odd. If \( \dim(V) > 1 \), then, \( V_\phi(G) \) does not have an \( o \)-basis. Also, for each \( \sigma \in C, e^\phi_{\gamma_\sigma} \neq 0 \) if and only if \( \frac{ht_\sigma}{l} \in \mathbb{Z} \).

**Proof.** By Proposition 3.3, it is enough to focus on \( V_\phi^o(C) \). Also, in the proof of Proposition 3.3, we have \( V_\phi^o(C) = \langle e^\phi_{\gamma_\sigma}, j = 1, 2, \ldots, t_\gamma \rangle \), where \( \sigma_j = a^{lp^k} \) and \( t_\gamma = \frac{|C|}{|C_{\gamma_\sigma}|} \). By (5.3) and the character tables, we compute that, for even \( i, j \) such that \( 1 \leq i, j \leq t_\gamma \),
\[
\langle e^\phi_{\gamma_\sigma}, e^\phi_{\gamma_\sigma} \rangle = 0 \iff \sum_{g \in G, \phi} \phi(g) \sigma_j (g) \overline{\phi(g) \sigma_j (g)} = 0 \iff \sum_{k=0}^{t_\gamma-1} \phi(a^{(k+j)p^k}) \overline{\phi(a^{(k+j)p^k})} = 0 \iff 2 \sum_{k=0}^{t_\gamma-1} \left[ 2 \cos \left( \left(2k + i\right) \frac{2\pi}{p^k} \right) \cos \left( (2k + i + j) \frac{2\pi}{p^k} \right) + 2 \sum_{k=0}^{t_\gamma-1} \left[ 2 \cos \left( (i-j) \frac{2\pi}{p^k} \right) - \sum_{k=0}^{t_\gamma-1} \cos \left( (i-j) \frac{2\pi}{p^k} \right) \right] \right] = 0 \iff 2l \cos \left( (i-j) \frac{2\pi}{p^k} \right) = 0 \quad \text{(since } \frac{2\pi}{p^k} \notin \mathbb{Z} \text{)}.
\]

Similar arguments work well for the remaining cases. Thus, we can conclude that
\[
\langle e^\phi_{\gamma_\sigma}, e^\phi_{\gamma_\sigma} \rangle = 0 \iff \begin{cases} \cos \left( (i-j) \frac{2\pi}{p^k} \right) = 0, & \text{if } i, j \text{ are both even or both odd;} \\ \sin \left( (i-j) \frac{2\pi}{p^k} \right) = 0, & \text{if otherwise.} \end{cases} \quad (7.2)
\]

We consider \( \gamma = (1, 2, 2, \ldots, 2) \in \Gamma_{\dim(V)}^{4n} \). Since \( \dim(V) > 1, \gamma \in \Delta \) and it is not hard to see that \( G_\gamma = \{ e \} \). So, \( t_\gamma = 1 \) and then \( \frac{ht_\sigma}{l} \in \mathbb{Z} \). By Proposition 4.4, \( \dim(V_\phi^o(C)) = 4 \). Thus, if \( V_\phi^o(C) \) has an \( o \)-basis, then there exist distinct \( 1 \leq i_1, i_2, i_3, i_4 \leq t_\gamma \) such that \( \{ e^\phi_{\gamma_{i_1}}, e^\phi_{\gamma_{i_2}}, e^\phi_{\gamma_{i_3}}, e^\phi_{\gamma_{i_4}} \} \) forms an \( o \)-basis. Since \( h \) is odd and \( 4 \mid l, \nu \left( \frac{2\pi}{p^k} \right) = -k \), for some positive integer \( k \). Hence, if there are at least three of \( i_1, i_2, i_3, i_4 \) which are all even or all odd, say \( i_1, i_2, i_3, \) then, by (7.2), there must exist odd integers \( a_1, a_2, a_3 \) such that
\[
i_1 - i_2 = a_1 \cdot 2^{k-1}, \quad i_1 - i_3 = a_2 \cdot 2^{k-1}, \quad \text{and } i_2 - i_3 = a_3 \cdot 2^{k-1}.
\]
This implies that $o_2 - o_3 = o_1$, which is a contradiction. If there are exactly two of $i_1, i_2, i_3, i_4$ which are all odd, say $i_1, i_2$, then, by (7.2), there must exist integer $s$ such that $i_1 - i_3 = s \cdot 2^k$. This implies that $i_1 = i_3 + s \cdot 2^k$ is even (because $i_3$ is even), which is a contradiction. Therefore, $V_\gamma^\phi(C)$ does not have an $o$-basis and by Proposition 5.2 we finish the proof for the first statement. The second statements is a consequence of Proposition 7.4 and a direct calculation as in Proposition 4.4.

Acknowledgements. The authors would like to thank Professor Bryan L. Shader and an anonymous referee for valuable comments, and Professor Hjalmar Rosengren for reviewing earlier drafts very carefully. The second author would like to thank Naresuan University for financial support on the project R2558C240.

REFERENCES

[17] B. Sambale. Blocks with defect group $Q_{2n} \times C_{2m}$ and $SD_{2n} \times C_{2m}$. Algebr. Represent. Theory, 16(6):1717–1732, 2013.
Orthogonal Bases of Brauer Symmetry Classes of Tensors for Groups Having Cyclic Support


