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A COUNTEREXAMPLE TO A QUESTION OF BAPAT AND SUNDER\textsuperscript{*}

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Abstract. A counterexample to a question of Bapat and Sunder is presented.

Key words. Permanent, Hadamard product, Oppenheim’s inequality.

AMS subject classifications. 15A15.

1. Introduction. In [1], Bapat and Sunder raise the question of whether the inequality
\[ \text{per}(A \circ B) \leq \text{per}(A) \prod_{j=1}^{n} b_{jj} \]  
holds for positive semidefinite \( n \times n \) matrices \( A \) and \( B \). The quantity \( \text{per}(A) \) denotes the permanent of \( A \) and the notation \( A \circ B \) is for the Hadamard (entrywise) product of \( A \) and \( B \). This is the permanental version of Oppenheim’s inequality. It is the objective of this article to provide a counterexample. The question is related to two other questions:

- The permanent on top conjecture, recently disproved by Shchegel’nov [4] which would have implied [1] had it been true.
- The inequality \( \text{per}(A \circ B) \leq \text{per}(A)\text{per}(B) \) introduced by Chollet [2] and established in the case \( n = 3 \) by Gregorac and Hentzel [3]. This inequality would be a consequence of [1] had it been true. Chollet’s conjecture remains open. For a relatively recent discussion of Chollet’s conjecture, the reader may consult Zhang [5].

2. The counterexample. With \( n = 7 \), we take
\[ A = \begin{pmatrix}
1 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & 1 & \cos \left( \frac{\pi}{8} \right) & -\cos \left( \frac{\pi}{8} \right) & \cos \left( \frac{\pi}{8} \right) & -\cos \left( \frac{\pi}{8} \right) & \cos \left( \frac{\pi}{8} \right) \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \cos \left( \frac{\pi}{8} \right) & -\cos \left( \frac{\pi}{8} \right) & 1 & \cos \left( \frac{\pi}{8} \right) & -\cos \left( \frac{\pi}{8} \right) \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\cos \left( \frac{\pi}{8} \right) & \cos \left( \frac{\pi}{8} \right) & -\cos \left( \frac{\pi}{8} \right) & 1 & \cos \left( \frac{\pi}{8} \right) \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \cos \left( \frac{\pi}{8} \right) & -\cos \left( \frac{\pi}{8} \right) & 1 & \cos \left( \frac{\pi}{8} \right) & -\cos \left( \frac{\pi}{8} \right) \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\cos \left( \frac{\pi}{8} \right) & \cos \left( \frac{\pi}{8} \right) & -\cos \left( \frac{\pi}{8} \right) & 1 & \cos \left( \frac{\pi}{8} \right) \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \cos \left( \frac{\pi}{8} \right) & -\cos \left( \frac{\pi}{8} \right) & 1 & \cos \left( \frac{\pi}{8} \right) & -\cos \left( \frac{\pi}{8} \right)
\end{pmatrix} \]

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and \( B = A^T \). Then it is easy to check that \( A \) and \( B \) are hermitian positive semidefinite matrices of rank two (with eigenvalue \( \frac{7}{2} \) of multiplicity two) and that

\[
\text{per}(A \circ B) = \frac{6185}{128},
\]

that \( \prod_{j=1}^7 b_{jj} = 1 \) and that \( \text{per}(A) = 45 \). We find that

\[
\frac{\text{per}(A \circ B)}{\text{per}(A) \prod_{j=1}^7 b_{jj}} = \frac{1237}{1152} > 1.
\]

REFERENCES