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ON THE AMPLITUDE AND PHASE RESPONSE IN THE NON-ABELIAN FOURIER TRANSFORM∗

PETER ZIZLER†

Abstract. In the context of the non-abelian Fourier transform, the natural extension of the amplitude and phase response to a convolution by a given filter mask are shown to be the polar decompositions of the Fourier transform matrices. A specific example regarding amplitude and phase response of a certain filter mask over the symmetric group $S_3$ is given.

Key words. Non-Abelian Fourier transform, Polar form of a matrix, Irreducible representation, Irreducible character, Amplitude and phase response, $G$-Circulant matrix.

AMS subject classifications. 15A18, 43A32, 42C99.

1. Introduction. Frequency filtering over a finite cyclic group is a classic and rich subject blending various areas of linear algebra and engineering. The complex exponentials become the eigenfunctions for any convolution operator regardless of the filter mask and the Fourier coefficients become the corresponding eigenvalues. Given a convolution filter mask we can study the frequency responses to the corresponding convolution operator, in particular, we can study the effects of the convolution operator on the individual complex exponentials. There is a vast literature on this classical subject, we refer the reader to [9] for a nice exposition.

The case of a convolution operator over a finite non-abelian group $G$ provides a slightly different paradigm. We have the notion of a Fourier transform over a non-abelian group $G$. This subject has been studied extensively, and to name few, we refer the reader to [5, 6, 8, 10, 11, 13, 14, 16, 17]. In this paper, we are interested in what the frequency response would mean to a $G$-convolution by a filter mask $\psi$ over a finite non-abelian group $G$.

Let $\mathbb{C}^n$ denote the $n$-dimensional vector space over the complex numbers. The standard basis for $\mathbb{C}^n$ is identified with the ordered group elements of $G$. 

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The $G$-convolution of $\psi$ and $\phi$ is the function
\[(\psi \ast \phi)(\sigma) = \sum_{\tau \in G} \psi(\sigma \tau^{-1}) \phi(\tau).\]
The action of $\psi$ on $\phi$ through $G$-convolution is captured by the matrix multiplication by the $G$-circulant matrix $C_G(\psi)$, in particular
\[\psi \ast \phi = C_G(\psi) \phi.\]
The $G$-circulant matrix $C_G(\psi)$ can be seen as the generalization of the (cyclic) circulant matrix to the non-abelian case. However, there are few important differences. In the cyclic case the eigenvectors of $C_G(\psi)$ are independent of $\psi$. This is no longer true, in general, in the non-abelian case. Moreover, the matrix $C_G(\psi)$ is usually non-normal. For more details on this topic, following this approach, we refer the reader to [18].

A function $\phi : \mathbb{C}^n \to \mathbb{C}$ is called a multiplicative character if
\[\phi(\sigma \tau) = \phi(\sigma) \phi(\tau) \text{ for all } \sigma, \tau \in G.\]
In the abelian case, the multiplicative characters are the eigenfunctions of the convolution operator. Observe that in the cyclic case we have as many multiplicative characters as there are group elements, $n$. We refer the reader to [3] for more details. In the non-abelian case the multiplicative characters are still eigenfunctions of the $G$-convolution operator. However, we do not have enough multiplicative characters when the group $G$ is non-abelian. Multiplicative characters must be constant on the conjugacy classes of $G$.

A finite dimensional representation of a finite group $G$ is a group homomorphism
\[\rho : G \mapsto GL(j, \mathbb{C}),\]
where $GL(j, \mathbb{C})$ denotes the general linear group of degree $j$, the set of all $j \times j$ invertible matrices. We refer to $j$ as the degree of the group representation.

Two group representations
\[\rho_1 : G \mapsto GL(j, \mathbb{C}) \text{ and } \rho_2 : G \mapsto GL(j, \mathbb{C})\]
are said to be equivalent if there exists an invertible matrix $T \in M_{j \times j}(F)$ such that
\[T \circ \rho_1(g) \circ T^{-1} = \rho_2(g)\]
for all $g \in G$. An irreducible group representation is a group representation for which there is no non-trivial subspace $W$ of $\mathbb{C}^j$ for which
\[\rho(g)W \subset W\]
for all \( g \in G \). Every finite dimensional group representation is equivalent to a representation by unitary matrices, see \([16]\) for example. For more information on group representations see \([4, 7, 15]\) for example.

Let \( \rho \) be any group representation of \( G \) (possibly reducible) of degree \( j \). Then we say that \( U \subset \mathbb{C}^j \) is \( G \)-stable if

\[
\rho(g)u \in U \quad \text{for all } g \in G \text{ and all } u \in U.
\]

Note that if \( \rho \) is also irreducible and \( U \) is \( G \)-stable then \( U = \{0\} \) or \( U = \mathbb{C}^j \), see \([4]\) for more details.

Let \( \widehat{G} \) be the set of all (equivalence classes) of irreducible representations of the group \( G \). Let \( \rho \in \widehat{G} \) be of degree \( j \) and let \( \phi \in \mathbb{C}^n \). Then the Fourier transform of \( \phi \) at \( \rho \) is the \( j \times j \) matrix

\[
\hat{\phi}(\rho) = \sum_{s \in G} \phi(s) \rho(s).
\]

The Fourier inversion formula, \( s \in G \), is given by

\[
\phi(s) = \frac{1}{|G|} \sum_{\rho_j \in \widehat{G}} d_j \text{tr} \left( \rho_j(s^{-1}) \hat{\phi} (\rho_j) \right).
\]

We alert the reader to an involution switch \( s \to s^{-1} \) in the summand functions. We refer the reader to \([12, 16, 17]\) for more details. Let \( \mathbb{C}[G] \) be the algebra of complex valued functions on \( G \) with respect to \( G \)-convolution. Let \( \psi = [c_0, c_1, \ldots, c_{n-1}]^T \in \mathbb{C}^n \) and identify the function \( \psi \) with its symbol

\[
\Psi = c_0 1 + c_1 g_1 + \cdots + c_{n-1} g_{n-1} \in \mathbb{C}[G].
\]

Let \( \psi \) and \( \phi \) be two elements in \( \mathbb{C}^n \) with the corresponding \( \Psi, \Phi \in \mathbb{C}[G] \). We have a natural identification

\[
\psi \star \phi \mapsto \Psi \Phi,
\]

where \( \Psi \Phi \) denotes multiplication in \( \mathbb{C}[G] \). For a reference on this and related results in representation theory see \([1]\), for example. The character of a group representation \( \rho \) is the complex valued function \( \chi : G \to \mathbb{C} \) defined by \( \chi(g) = \text{tr}(\rho(g)) \).

For all \( g \in G \), the quantity \( \chi(g) \) is a sum of complex roots of unity. Moreover, we have \( \chi(g^{-1}) = \chi(g) \) for all \( g \in G \), see \([4]\) for example. A character is called irreducible if the underlying group representation is irreducible. We define an inner product on the space of class functions, functions on \( G \) that are constant its conjugacy classes, as

\[
\langle \chi, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\theta(g)}.
\]
Note that a character is a class function. We have as many irreducible characters as there are conjugacy classes of $G$. If $G$ is abelian, then we have $n$ irreducible characters.

With respect to the usual inner product we have

$$\langle \chi_i, \chi_j \rangle = \delta_{ij},$$

where $\delta_{i,j}$ is the Kronecker delta. Irreducible characters form a basis for the space of class functions on $G$, see [4] for example. For further references on this subject, we refer the reader to [2, 16, 17].

The Fourier transform gives us a natural isomorphism

$$\mathbb{C}[G] \Rightarrow M(\hat{G}),$$

where

$$M(\hat{G}) = M_{d_1 \times d_1}(\mathbb{C}) \oplus M_{d_2 \times d_2}(\mathbb{C}) \oplus \cdots \oplus M_{d_r \times d_r}(\mathbb{C})$$

with $d_1^2 + d_2^2 + \cdots + d_r^2 = n$. A typical element of $\mathbb{C}^n$ is a complex valued function

$$\psi = [c_0, c_1, \ldots, c_{n-1}]^T$$

and the typical element of $M(\hat{G})$ is the direct sum of Fourier transforms

$$\hat{\phi}(\rho_1) \oplus \hat{\phi}(\rho_2) \oplus \cdots \oplus \hat{\phi}(\rho_r).$$

The Fourier transform turns convolution into (matrix) multiplication

$$\widehat{\psi \star \phi} = \bigoplus_{j=1}^r \psi_j^* \phi_j = \hat{\psi} \hat{\phi}$$

We equip $M(\hat{G})$ with an inner product as follows. Let $v = \hat{\phi}(\rho_1) \oplus \hat{\phi}(\rho_2) \oplus \cdots \oplus \hat{\phi}(\rho_r)$ and $w = \hat{\zeta}(\rho_1) \oplus \hat{\zeta}(\rho_2) \oplus \cdots \oplus \hat{\zeta}(\rho_r)$. Then

$$(v \cdot_F w) = \frac{d_1}{|G|} \text{tr} \left( \hat{\phi}(\rho_1) \hat{\zeta}^*(\rho_1) \right) + \frac{d_2}{|G|} \text{tr} \left( \hat{\phi}(\rho_2) \hat{\zeta}^*(\rho_2) \right) + \cdots + \frac{d_r}{|G|} \text{tr} \left( \hat{\phi}(\rho_r) \hat{\zeta}^*(\rho_r) \right),$$

where $\hat{\zeta}^*(\rho)$ denotes the adjoint of $\hat{\zeta}(\rho)$. The Fourier transform is a unitary transformation from $\mathbb{C}^n$ onto $\left(M(\hat{G}), \cdot_F \right)$, see [18] for more details. We say that two vectors $\psi, \phi$ in $\mathbb{C}^n$ are $G$-decorrelated if

$$\sum_{t \in G} \overline{\psi}(t) \phi(t \tau) = 0 \text{ for all } \tau \in G.$$
Let $\rho_j$ be a matrix corresponding an irreducible representation of the group $G$ and let $\rho_j(k, l)$ be its $(k, l)$ entry. Consider the function $\rho_j(k, l)(g^{-1}) \in \mathbb{C}^n$ and, by abusing notation slightly for simplicity, denote this function by $\rho_j(k, l)$. The set of functions

$$\left\{ \sqrt{\frac{d_j}{|G|}} \rho_j(k, l) \mid k, l \in 1, \ldots, d_j \right\}$$

has $n$ elements and this set is orthonormal. Furthermore, this set has the further property of $G$-decorrelation, in particular, the functions in this set are mutually $G$-decorrelated. Note that if $d_j = 1$ then the corresponding $\rho_j$ is the corresponding irreducible character, automatically a multiplicative character and an eigenfunction of the $G$-convolution operator. The following is taken from [19].

**Theorem 1.1.** Let $\phi \in \mathbb{C}^n$ and let $\{\rho_j\}_{j=1,\ldots,r}$ be the set of all irreducible representations of $G$, $\rho_j$ has size $d_j$. Let $\rho_j(k, l)$ be the $(k, l)$ entry in the $d_j \times d_j$ matrix of $\rho_j$. Fix $j, k, l$ and consider the (involuted) function $\rho_j(k, l)(g^{-1}) \in \mathbb{C}^n$ and denote this function by $\rho_j(k, l)$. Then $\phi$ can be written as a $G$-decorrelated sum

$$\phi = \sum_{j=1}^{n} \sum_{k,l=1}^{d_j} \phi_j(k, l)$$

with

$$\phi_j(k, l) = \frac{d_j}{|G|} \langle \phi, \rho_j(k, l) \rangle \rho_j(k, l).$$

**2. Main results.** The above remarks indicate that the set of functions

$$\{\rho_j(k, l)\}_{j=1,\ldots,r}$$

could qualify as the generalization of the frequencies (complex exponentials) from the cyclic case. The main problem, however, lies in the fact that the individual function $\rho_j(k, l)$ is typically not an eigenfunction of the $G$-convolution operator. On the other hand, the eigenfunctions of the $G$-circulant matrices are not independent of the convolution filter mask $\psi$. A solution to this is to look at the following ($\psi$ independent) subspace of $\mathbb{C}^n$

$$\text{Im}(P_j) = \text{span}\{\rho_j(k, l) \mid k, l \in 1, \ldots, d_j\},$$

where $P_j$ is the projection operator given by

$$P_j(\phi) = \phi_j.$$
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with

$$\phi_j(s) = \frac{d_j}{|G|} \text{tr} \left( \rho_j(s^{-1}) \hat{\varphi}(\rho_j) \right).$$

The action of the linear operator $P_j$ in the Fourier domain is given by the (matrix) multiplication by the vector

$$0 \oplus \cdots \oplus 0 \oplus I_j \oplus 0 \oplus \cdots \oplus 0,$

where the $d_j \times d_j$ identity matrix $I_j$ is in the $j$th position.

To understand the frequency response to a $G$-convolution by a filter mask $\psi$ over a non-abelian group $G$ we look at the subspace $\text{Im}(P_j) \subset \mathbb{C}^n$. The subspace $\text{Im}(P_j)$ is $C_G(\psi)$ invariant for all $\psi \in \mathbb{C}^n$. In the case of a cyclic group $G$ this invariant subspace is spanned by the corresponding complex exponential (frequency) and the restriction of $C_G(\psi)$ to this subspace yields the corresponding Fourier coefficient. Therefore, to generalize to the non-abelian setting, we consider the matrix $C_G(\psi)$ restricted to the subspace $\text{Im}(P_j)$.

However, the subspace $\text{Im}(P_j)$ with dimension equal to $d_j^2$ is unnecessarily too large for the frequency response analysis. In what follows, we will identify a $d_j$-dimensional subspace $M_{j,k}$ of $\text{Im}(P_j)$ that we will prove is also $C_G(\psi)$ invariant regardless of the filter mask $\psi$. We will show this is the smallest subspace with this property.

**Lemma 2.1.** Consider the function $\rho_j(k,l)$ for fixed $j, k, l$. Then

$$C_G(\psi) \left( \rho_j(k,l) \right) = \sum_{i=1}^{d_j} \hat{\psi}_j(i,l) \rho_j(k,i).$$

**Proof.** The key observation for the proof is the following identity

$$\hat{\rho}_j(k,l) = 0 \oplus \cdots \oplus 0 \oplus \frac{|G|}{d_j} E_j(l,k) \oplus 0 \oplus \cdots \oplus 0,$$

where $E_j(l,k)$ is the $d_j \times d_j$ matrix whose entries are all zero except at the location $(l,k)$, where the matrix entry is equal to 1. The matrix $E_j(l,k)$ is in the $j$th position in the direct sum. This can be seen from the expression for the inverse Fourier transform

$$\rho_j(k,l) = \text{tr} \left( \rho_j(s^{-1}) E_j(l,k) \right).$$

Now we observe

$$C_G(\psi) \left( \rho_j(k,l) \right) = \text{tr} \left( \rho_j(s^{-1}) \hat{\psi}(\rho_j) E_j(l,k) \right) = \sum_{i=1}^{d_j} \hat{\psi}_j(i,l) \rho_j(k,i).$$
An important feature here is the fact that the image of any \( \rho_j(k,l) \), under \( C_G(\psi) \), does not involve all the functions \( \rho_j(k,l), k,l \in \{1,\ldots,d_j\} \), but only the set \( \{\rho_j(k,l) | l \in \{1,\ldots,d_j\}\} \). Define a subspace

\[
M_{j,k} = \text{span}\{\rho_j(k,l) | l \in \{1,\ldots,d_j\}\}
\]

and observe the spaces \( \{M_{j,k}\}_{j \in \{1,\ldots,r\},k \in \{1,\ldots,d_j\}} \) are mutually orthogonal

\[
M_{j,k} \cap M_{r,s} = (0) \text{ for } (j,k) \neq (r,s)
\]

if \( d_j = 1 \) we write \( M_{j,1} = M_j \).

**Theorem 2.2.** Consider the subspace \( M_{j,k} \subset \text{Im}(P_j) \subset \mathbb{C}^n \) as above. Then we have the following:

1. The subspace \( M_{j,k} \) is an invariant subspace for \( C_G(\psi) \) for all \( \psi \in \mathbb{C}^n \).
2. The corresponding coefficients of the image \( C_G(\psi)(\rho_j(k,l)) \), with respect to the basis elements \( \{\rho_j(k,l) | l \in \{1,\ldots,d_j\}\} \), are the entries in the respective columns of the Fourier matrix \( \hat{\psi}_j \). In particular, if we fix \( j \), then with respect to the basis elements \( \{\rho_j(k,l) | l \in \{1,\ldots,d_j\}\} \) the action of \( C_G(\psi) \) on \( M_{j,k} \) is independent of \( k \).
3. If \( V \) is any non-zero subspace of \( M_{j,k} \) such that \( V \) is also an invariant subspace for \( C_G(\psi) \) for all \( \psi \in \mathbb{C}^n \) then \( V = M_{j,k} \).

**Proof.** The first two claims directly follow from Lemma 2.1. To address the claim 3 suppose we have a non-zero subspace \( V \subset M_{j,k} \) of dimension strictly less than \( d_j \). Suppose, furthermore, that \( V \) is also an invariant subspace for \( C_G(\psi) \) for all \( \psi \in \mathbb{C}^n \). This would make the subspace \( V \) a nontrivial \( G \)-stable subspace of \( M_{j,k} \) and hence a \( G \)-stable subspace of \( \mathbb{C}^j \) contradicting the irreducibility of the group representation \( \rho_j \).

Theorem 2.2 tells us the space \( M_{j,k} \) is the correct space to work on when analyzing the frequency response of a \( G \)-convolution by a filter mask \( \psi \). Recall the action of \( C_G(\psi) \) on \( M_{j,k} \) is identical for all \( k \), with respect to the corresponding basis, and is captured by the Fourier matrix \( \hat{\psi}_j \). Therefore, in the frequency response study for any given filter mask \( \psi \) we can choose any \( k_0 \in \{1,\ldots,d_j\} \) and set \( M = M_{j,k_0} \).

We have an orthogonal subspace decomposition of \( \mathbb{C}^n \) given by

\[
\mathbb{C}^n = \bigoplus M_{j,k} \text{ with } j \in \{1,\ldots,r\}, k \in \{1,\ldots,d_j\}.
\]

Each subspace \( M_{j,k} \) is of dimension \( d_j \) and is an invariant subspace for \( C_G(\psi) \) for any filter mask \( \psi \in \mathbb{C}^n \). This allows us to proceed along the following lines. Let \( \psi \in \mathbb{C}^n \) and let \( \hat{\psi}_j = U_j Q_j \) be the polar decomposition of the Fourier matrix \( \hat{\psi}_j \).
with \( Q_j = \sqrt{\hat{\psi}_j^* \hat{\psi}_j} \) and \( U_j \) being unitary. Then the amplitude response of the \( G \)-convolution by \( \psi \) is the set of matrices \( \{ Q_j \}_{j=1,...,r} \) and the phase response of the \( G \)-convolution by \( \psi \) is the set of matrices \( \{ U_j \}_{j=1,...,r} \).

Note that we abuse the terminology slightly here when it comes to the phase response, as the phase response in the cyclic case refers to the angle \( \theta \) as opposed to \( \exp(i\theta) \). Here we would work with \( \exp(i\theta) \) instead. The magnitude of the amplitude response can be defined by the following set, \( ||.||_F \) denotes the Frobenius norm,

\[
\{ ||Q_j||_F \}_{j=1,...,r}.
\]

Each phase response matrix \( U_j \) can have \( d_j \) values associated with it that are \( \{ \text{Arg}(\lambda_k) \}_{k \in 1,...,d_j} \), where \( \lambda_k \) is an eigenvalue of \( U_j \) and \( \text{Arg}(\cdot) \) stands for the argument of a complex number.

To obtain a filter mask \( \psi_A \) that has the same amplitude response as the given filter mask \( \psi \) but trivial phase response we set

\[
\psi_A(s) = \frac{1}{|G|} \sum_{\rho_j \in \hat{G}} d_j \text{tr} (\rho_j(s^{-1})Q_j).
\]

Similarly, to obtain a filter mask \( \psi_P \) that has the same phase response as the given filter mask \( \psi \) but trivial amplitude response we set

\[
\psi_P(s) = \frac{1}{|G|} \sum_{\rho_j \in \hat{G}} d_j \text{tr} (\rho_j(s^{-1})U_j).
\]

**APPLICATION.** The following can be seen as an application of the above decomposition. Let \( \psi \in \mathbb{C}^n \) be a filter mask and let \( \phi \in \mathbb{C}^n \). We can decompose the filtering action by \( \psi \) on \( \phi \) as

\[
\psi * \phi = \psi_P * (\psi_A * \phi) = (\psi_P * \psi_A) * \phi,
\]

where \( ||\psi_P * \phi||_2 = ||\phi||_2 \) and \( ||\psi_A * \phi||_2 = ||\psi * \phi||_2 \). Equivalently, we obtain the following matrix factorization

\[
\]

In the cyclic case, the above matrix factorization is the polar decomposition of the matrix \( G_G(\psi) \). This can be readily seen from the singular value decomposition of this matrix. The matrix \( G_G(\psi) \) is a normal matrix and its eigenvectors, the complex exponentials, are \( \psi \) independent. However, the above matrix factorization is no longer the polar decomposition of \( G_G(\psi) \), in general, for non-abelian groups \( G \).
3. The case of the symmetric group $S_3$. We will consider the symmetric group $S_3$ in our example. The group $G = S_3$ consists of elements

$$g_0 = (1); \quad g_1 = (12); \quad g_2 = (13); \quad g_3 = (23); \quad g_4 = (123); \quad g_5 = (132).$$

Let $\psi = [c_0, c_1, c_2, c_3, c_4, c_5]^T$, then the $G$-convolution by a function $\psi \in \mathbb{C}^n$ can be induced by a $G$-circulant matrix $C_G(\psi)$ given by

$$C_G(\psi) = \begin{pmatrix}
  c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\
  c_1 & c_0 & c_4 & c_5 & c_2 & c_3 \\
  c_2 & c_5 & c_0 & c_1 & c_3 & c_1 \\
  c_3 & c_4 & c_5 & c_0 & c_1 & c_2 \\
  c_4 & c_3 & c_1 & c_2 & c_5 & c_0 \\
  c_5 & c_2 & c_3 & c_1 & c_0 & c_4
\end{pmatrix}.$$}

The group $S_3$ has three conjugacy classes

$$\{g_0\}, \quad \{g_1, g_2, g_3\}, \quad \{g_4, g_5\}.$$

We have three irreducible representations, two of which are one dimensional, $\rho_1$ is the identity map, $\rho_2$ is the map that assigns the value of 1 if the permutation is even and the value of −1 if the permutation is odd. Finally, we have $\rho_3$, the two dimensional irreducible representation of $S_3$, defined by the following assignment

$$g_0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad g_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad g_2 \mapsto \begin{pmatrix} 0 & e^{2\pi i/3} \\ e^{-2\pi i/3} & 0 \end{pmatrix};$$

$$g_3 \mapsto \begin{pmatrix} 0 & e^{-2\pi i/3} \\ e^{2\pi i/3} & 0 \end{pmatrix}; \quad g_4 \mapsto \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{pmatrix}; \quad g_5 \mapsto \begin{pmatrix} e^{-2\pi i/3} & 0 \\ 0 & e^{2\pi i/3} \end{pmatrix}.$$

The irreducible characters of $S_3$ are given by

$$\chi_1 = (1, 1, 1, 1, 1)^T,$$

$$\chi_2 = (1, -1, -1, 1, 1)^T,$$

$$\chi_3 = (2, 0, 0, -1, -1)^T,$$

where $\chi_1$ and $\chi_2$ are also multiplicative characters. We record

$$\rho_1(1,1) = (1,0,0,0,e^{-2\pi i/3},e^{2\pi i/3})^T,$$

$$\rho_2(1,2) = (0,1,e^{-2\pi i/3},e^{2\pi i/3},0,0)^T,$$

$$\rho_3(2,1) = (0,1,e^{2\pi i/3},e^{-2\pi i/3},0,0)^T,$$

$$\rho_3(2,2) = (1,0,0,0,e^{2\pi i/3},e^{-2\pi i/3})^T.$$
and observe
\[
M_1 = [1, 1, 1, 1, 1]^T,
M_2 = [1, -1, -1, -1, 1]^T,
M_{3,1} = \text{span}\{\rho_3(1, 1), \rho_3(1, 2)\},
M_{3,2} = \text{span}\{\rho_3(2, 1), \rho_3(2, 2)\}.
\]

Let us consider \( \psi = [2, 3, 1, 0, 4, 5]^T \) and analyze the frequency response of the \( S_3 \)-convolution by the filter mask \( \psi \). We calculate the corresponding amplitude and phase response to this convolution. We have
\[
\hat{\psi}_1 = 15; \quad \hat{\psi}_2 = 7; \quad \hat{\psi}_3 = \begin{pmatrix} -\frac{5}{2} - \frac{\sqrt{3}}{2}i & \frac{5}{2} + \frac{\sqrt{3}}{2}i \\ -\frac{5}{2} - \frac{\sqrt{3}}{2}i & -\frac{5}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix}.
\]

We obtain
\[
\hat{\psi}_3 = U_3Q_3,
\]
where
\[
U_3 = \begin{pmatrix} -0.96 - 0.28i & -0.93 + 0.38i \\ -0.01 + 0.05i & -0.02 + 0.05i \end{pmatrix} \quad \text{and} \quad Q_3 = \begin{pmatrix} 2.65 & -2.65 \\ -2.65 & 2.65 \end{pmatrix}.
\]

The eigenvalues of \( Q_3 \) are 0 and 5.29 and thus \( ||Q_3||_F = 5.29 \). The eigenvalues of \( U_3 \) are \( \lambda_1 = -0.96 - 0.28i \) and \( \lambda_2 = -0.93 + 0.38i \). Note that \( \text{Arg}(\lambda_1) = -163.67^\circ \) and \( \text{Arg}(\lambda_2) = 157.67^\circ \).

The amplitude response of \( S_3 \)-convolution by the filter mask \( \psi \) is the sequence
\[
\begin{pmatrix} 15, 7, \begin{pmatrix} 2.65 & -2.65 \\ -2.65 & 2.65 \end{pmatrix} \end{pmatrix}
\]
and the phase response of \( S_3 \)-convolution by the filter mask \( \psi \) is the sequence
\[
\begin{pmatrix} 1, 1, \begin{pmatrix} -0.96 - 0.28i & -0.93 + 0.38i \\ -0.01 + 0.05i & -0.02 + 0.05i \end{pmatrix} \end{pmatrix}
\]
To obtain a filter mask \( \psi_A \) that has the same amplitude response as \( \psi = [2, 3, 1, 0, 4, 5]^T \) but trivial phase response we set
\[
\psi_A(s) = \frac{1}{6} \left( 15\rho_1(s^{-1}) + 7\rho_2(s^{-1}) + 2\text{tr} \left( \rho_3(s^{-1}) \begin{pmatrix} 2.65 & -2.65 \\ -2.65 & 2.65 \end{pmatrix} \right) \right)
\]
and obtain
\[
\psi_A = [5.43, -0.43, 2.22, 2.22, 2.78, 2.78]^T.
\]
Similarly, to obtain a filter mask \( \psi_p \) that has the same phase response as \( \psi = [2, 3, 1, 0, 4, 5]^T \) but trivial amplitude response we set

\[
\psi_p(s) = \frac{1}{6} \left( 15 \rho_1(s^{-1}) + 7 \rho_2(s^{-1}) + 2 \text{tr} \left( \rho_3(s^{-1}) \begin{pmatrix} -0.96 - 0.28i & -0.01 + 0.05i \\ 0.02 + 0.05i & -0.93 + 0.38i \end{pmatrix} \right) \right)
\]

and obtain

\[
\psi_p = [3.04 + 0.03i, 1.34 + 0.03i, 1.33 - 0.01i, 1.33 - 0.03i, 3.79 - 0.01i, 4.17 - 0.03i]^T.
\]

REFERENCES


