Electronic Journal of Linear Algebra

Volume 32 *Volume 32 (2017)* Article 23

2017

Computation of State Reachable Points of Second Order Linear Time-Invariant Descriptor Systems

Subashish Datta Ph.D.
Indian Institute of Technology Mandi, Mandi, Himachal Pradesh, India-175005, sd@iitmandi.ac.in

V Mehrmann
Technische Universitaet Berlin, mehrmann@math.TU-Berlin.DE

Recommended Citation
Datta, Subashish Ph.D. and Mehrmann, V. (2017), "Computation of State Reachable Points of Second Order Linear Time-Invariant Descriptor Systems", *Electronic Journal of Linear Algebra*, Volume 32, pp. 301-316.
DOI: https://doi.org/10.13001/1081-3810.3303

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

COMPUTATION OF STATE REACHABLE POINTS OF SECOND ORDER LINEAR TIME-INVARIANT DESCRIPTOR SYSTEMS*

SUBASHISH DATTA† AND VOLKER MEHRMANN‡

Abstract. This paper considers the problem of computing the state reachable points (from the origin) of a linear constant coefficient second order descriptor system. A new method is proposed to compute the reachable set in a numerically stable way. The original descriptor system is transformed into a strangeness-free system within the behavioral framework followed by a projection that separates the system into different order differential and algebraic equations while keeping the original state variables. This reformulation is followed by a first order formulation that avoids all unnecessary smoothness requirements. For the resulting first order system, it is shown that the computation of the image space of two matrices, associated with the projected system, is enough to numerically compute the reachable set. Moreover, a characterization is presented of all the inputs by which one can reach an arbitrary point in the reachable set. These results are used to compute two different types of reachable sets for second order systems. The new approach is demonstrated through a numerical example.

Key words. Linear time-invariant descriptor system, Behavior formulation, Strangeness-free formulation, Reachability, Derivative array, Second order system.

AMS subject classifications. 93C05, 93C15, 93B05.

1. Introduction. Due to the wide availability of automated modeling tools [1, 18, 25] the modeling of dynamical systems via descriptor systems, where the system equation is a differential-algebraic equation (DAE), has become industrial standard in many application domains, see e.g. [4, 12, 21, 30, 33, 34].

Modeling with descriptor systems has the major advantage that constraints which restrict the dynamics of the system can be explicitly incorporated in the model. This is important, since they usually describe real physical properties like conservation laws, symmetries, or other invariants of the system. Further constraints may be implicitly hidden in the system that can only be revealed via further differentiation. This leads to a major distinction to ordinary state-space systems in the sense that descriptor systems not only integrate but also differentiate, see [21]. The number of required differentiations is usually described by an *index*, see [28] for a comparison of different index concepts. If in descriptor systems the inputs are not sufficiently smooth, then impulsive responses may arise [11, 12, 21, 32, 37] and a regularization process may be needed, see [10]. This is important, in particular, in the context of control systems, where typically the inputs are very often discontinuous, like in bang-bang control [2].

In this paper, we consider numerical methods to detect *reachability*, i.e., whether a control input exists which transfers from one state to another state in finite time, for second order linear descriptor systems of the form

$$M\ddot{x} + D\dot{x} + Kx = Bu, \tag{1.1}$$

*Received by the editors on May 11, 2016. Accepted for publication on June 12, 2017. Handling Editor: Michael Tsatsomeros. This research was supported by IMU Einstein Foundation Program and *Einstein Center ECMath of Einstein Stiftung Berlin*, within Research Center Matheon.

†School of Computing and Electrical Engineering, Indian Institute of Technology Mandi, Mandi, Himachal Pradesh, India-175005 (sd@iitmandi.ac.in).

‡Institut für Mathematik, MA 4-5, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, Germany (mehrmann@math.tu-berlin.de).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

where $M, D, K \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $x : \mathbb{R} \to \mathbb{R}^n$ is the state and $u : \mathbb{R} \to \mathbb{R}^m$ is an input (control) to the system. Furthermore, initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$ are assumed to be available. We assume that the system is *regular*, i.e., there exists a complex number s such that $s^2M + sD + K$ is invertible, but more general situations are possible, see [29, 35]. We introduce numerical methods to compute the set of all *state reachable points* via some chosen set of input functions. In particular, we show that the reachability sets are characterized by the image space of certain matrices.

The classical textbook approach to study these problems would be to turn the second order system into first order by introducing a new variable $v = \dot{x}$ and then treat the reachability questions for the first order case. But this is somewhat subtle for descriptor systems.

Example 1.1. Consider the second order descriptor system from [8],

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\ddot{x} + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\dot{x} + \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}x = \begin{bmatrix}b_1\\ b_2\end{bmatrix}u,$$

which has the solution $x_1 = b_2u$, $x_2 = b_1u - b_2(u + \dot{u} + \ddot{u})$. If $b_2 \neq 0$ and $\ddot{u}$ exists, that is, u is twice continuously differentiable, then the solution is continuous, but if we introduce a new variable $v = \dot{x}$ and transfer the second order system to first order, then the solution is

$$\begin{aligned} x_1 &= b_2u, \quad x_2 = b_1u - b_2\left(u + \dot{u} + \ddot{u}\right),\\ v_1 &= b_2\dot{u}, \quad v_2 = b_1\dot{u} - b_2\left(\dot{u} + \ddot{u} + u^{(3)}\right), \end{aligned}$$

and hence, for a continuous solution, u has to be three times continuously differentiable. One way to avoid this dilemma of extra smoothness requirements, and this is the approach that we follow below, is to only introduce $v_1 = b_2\dot{u}$ as a new variable, see [8], where this process is called *trimmed first order formulation*.

Another difficulty in second order systems is that one has to distinguish between the set of states that can be reached for the second order formulation and that of a first order formulation, see also [24].

Definition 1.2. Consider the second order descriptor system (1.1). A vector $x_1 \in \mathbb{R}^n$ is said to be *reachable from the initial condition* $x(t_0) = x_0$, if there exists a (sufficiently smooth) control input u, an initial value for the derivative $\dot{x}(t_0) = v(t_0)$, and a finite time $t_1 > 0$ such that $x(t_1) = x_1$.

A first order formulation of (1.1) with an extended state vector $\xi_1 := [\tilde{v}_1^T,\ x_1^T]^T \in \mathbb{R}^{\tilde{n}+n}$ is said to be *reachable from the initial condition* $\xi(t_0) := [\tilde{v}(t_0)^T,\ x(t_0)^T]^T \in \mathbb{R}^{\tilde{n}+n}$, if there exists a (sufficiently smooth) control input u and a finite time $t_1 > 0$ such that $\xi(t_1) = \xi_1$.

It is obvious that the two reachability sets are different.

Example 1.3. A simple second order control system is the model of a two-dimensional, linearized model of a three-link mobile manipulator from [20], see also [5, 10], which has the form

$$M_0\ddot{\phi} + D_0\dot{\phi} + K_0\phi = F_0^T\lambda + S_0u, \quad F_0\phi = 0,$$

where M_0, D_0, K_0 are mass, damping, and stiffness matrix, λ is a Lagrange multiplier, and $F_0^T\lambda$ is the generalized constraint force. Setting $x = \begin{bmatrix}\phi^T & \lambda^T\end{bmatrix}^T$, we obtain the second order descriptor system

$$\begin{bmatrix}M_0 & 0\\ 0 & 0\end{bmatrix}\ddot{x} + \begin{bmatrix}D_0 & 0\\ 0 & 0\end{bmatrix}\dot{x} + \begin{bmatrix}K_0 & -F_0^T\\ F_0 & 0\end{bmatrix}x = \begin{bmatrix}S_0\\ 0\end{bmatrix}u.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

Avoiding extra regularity one can use the trimmed first order formulation [8] and introduce $\nu = \dot{\phi}$. Since ϕ and ν are restricted by the cleaning surface, it makes a major difference whether one wants to reach a state ϕ_1 and does not care about the velocity ν or whether one wants to reach ϕ_1 with a given velocity ν_1.

The paper is organized as follows. In Section 2, we briefly recall the *derivative array* approach for first order systems to obtain a regularized model associated with the original descriptor system in which all constraints are explicitly available. Using the regularized system we define two projection matrices to separate the descriptor system additively into its differential and algebraic parts. Based on this decomposition, we derive a method to numerically compute the state reachable points of the descriptor system in Section 3. The results are then extended to second order systems in Section 4. A numerically stable regularization procedure is presented in Section 5, and based on this procedure, the reachable sets are characterized. The results are illustrated by a numerical example from mechanical multi-body systems.

2. Regularization of descriptor systems. A traditional way to carry out the analysis and regularization of descriptor systems is to transform the system into an ordinary state space system by resolving all the algebraic constraint equations. However, in this way, typically the variables may lose their physical interpretation and what is worse, the algebraic constraints are not available any longer in the system, and in numerical simulation or control, due to discretization and rounding errors, these constraints are violated, see the discussion in [10, 21, 22]. This approach gives particularly bad results when the solution of the algebraic equations is sensitive to perturbations.

Another classical approach, presented, e.g. in [12, 21, 27], is to use canonical forms to identify the invariants and to decouple the original system into the fast (algebraic) and slow (differential) subsystem. But the transformation to these canonical forms may be arbitrarily ill-conditioned [5, 21], so that small perturbations lead to physically unrealistic subsystems. But since the canonical forms are typically not numerically computable, this approach is of a more theoretical value. Partially, these difficulties can be avoided using staircase forms [15, 36] under orthogonal transformations, see [5] for a survey. However, also the computation of these staircase forms is very subtle in finite precision arithmetic, a complete error analysis is not available, and the physical interpretation of variables is changed.

To avoid the described difficulties, many approaches have been suggested for index reduction and reformulation of differential-algebraic systems, see [21, 23]. A very successful procedure in practice is the *derivative array approach*, see [9, 10, 21]. In this approach, since derivatives are needed, one adds a sufficient number of derivatives of the original model equations to the system, so that all necessary derivative information is available. From the resulting over-determined system then, via orthogonal projections, one identifies and separates the differential and algebraic equations. The derivative array approach is very robust to perturbations, and it has been implemented successfully in numerical simulation codes for linear and nonlinear differential-algebraic systems [21]. Another surplus of this approach is that the physical meaning of all the variables is preserved.

Let us briefly recall the derivative array approach for first order systems of the form

$$E\dot{x} = Ax + Bu, \quad x(t_0) = x_0, \tag{2.2}$$

where $E \in \mathbb{R}^{n,n}$, $A \in \mathbb{R}^{n,n}$, and $B \in \mathbb{R}^{n,m}$, $x : \mathbb{R} \to \mathbb{R}^n$ is the state and $u : \mathbb{R} \to \mathbb{R}^m$ is the input (control). Assuming that the system is regular, i.e., there exists a complex number s such that $sE - A$ is invertible, see [10] for the general case, we express the state equation in a behavior framework, see [31], i.e., rewrite

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

(2.2) as

$$\mathbf{E}\dot{z} = \mathbf{A}z, \tag{2.3}$$

where $\mathbf{E} := [E \;\; 0] \in \mathbb{R}^{n,(n+m)}$ and $\mathbf{A} := [A \;\; B] \in \mathbb{R}^{n,(n+m)}$, by introducing $z = \begin{bmatrix} x^T & u^T \end{bmatrix}^T$. By performing a sequence of differentiations of (2.3) and stacking the original system and its derivatives up to order μ on top of each other, we get a *derivative array* $\mathcal{M}_\mu \dot{z}_\mu = \mathcal{N}_\mu z_\mu$, where

$$\mathcal{M}_\mu = \begin{bmatrix} \mathbf{E} & 0 & \cdots & 0 \\ -\mathbf{A} & \mathbf{E} & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots\cdots & -\mathbf{A} & \mathbf{E} \end{bmatrix}, \quad \mathcal{N}_\mu = \begin{bmatrix} \mathbf{A} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad z_\mu = \begin{bmatrix} z \\ \dot{z} \\ \vdots \\ z^{(\mu)} \end{bmatrix}.$$

To obtain a regularized system we use the following theorem, see [21].

THEOREM 2.1. *There exist integers μ, a, and d such that the inflated pair $(\mathcal{M}_\mu, \mathcal{N}_\mu)$ associated with the given pair $(\mathbf{E}, \mathbf{A})$ has the following properties:*

1. $\operatorname{rank}\mathcal{M}_\mu = (\mu+1)n - a$ *such that there exists a matrix Z_2 with orthogonal columns of size $((\mu+1)n, a)$ satisfying $Z_2^T \mathcal{M}_\mu = 0$.*
2. $\operatorname{rank}A_2 = a$, *where $A_2 = Z_2^T \mathcal{N}_\mu [I_n \; 0 \; \cdots \; 0]^T$ such that there exists a matrix T_2 with orthogonal columns of size (n, d), $d = n - a$ satisfying $A_2 T_2 = 0$.*
3. $\operatorname{rank}\mathbf{E}T_2 = d$ *such that there exists a matrix Z_1 with orthogonal columns of size (n, d) satisfying $\operatorname{rank}E_1 T_2 = d$ with $E_1 = Z_1^T \mathbf{E}$.*

The smallest $\mu \geq 0$ for which these conditions hold is called the *strangeness-index* of (2.3). Using the orthogonal matrices Z_1, Z_2 and T_2, we obtain a strangeness-free ($\mu = 0$) system in behavior form associated with (2.3) given by

$$\begin{bmatrix} \widehat{E}_1 \\ 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \widehat{A}_1 \\ \widehat{A}_2 \end{bmatrix} z, \tag{2.4}$$

where the coefficients are given by

$$\widehat{E}_1 = Z_1^T \mathbf{E}, \quad \widehat{A}_1 = Z_1^T \mathbf{A}, \quad \widehat{A}_2 = Z_2^T \mathcal{N}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}.$$

Systems (2.3) and (2.4) have the same solution set in the behavior sense and can be represented as $\mathcal{E}\dot{x} = \mathcal{A}x + \mathcal{B}u$, or

$$E_1 \dot{x} = A_1 x + B_1 u, \tag{2.5a}$$
$$0 = A_2 x + B_2 u, \tag{2.5b}$$

respectively, where $E_1 = Z_1^T E \in \mathbb{R}^{d,n}$ has full row rank,

$$A_1 = \widehat{A}_1 \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad A_2 = \widehat{A}_2 \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad B_1 = \widehat{A}_1, \quad B_2 = \widehat{A}_2 \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

Note that (2.5) may not be strangeness-free as a free system with $u = 0$, but it satisfies $\operatorname{rank}[\mathcal{E}, \mathcal{A}S_\infty, \mathcal{B}] = n$, where the columns of the matrix S_∞ span the kernel of $\mathcal{E}$, i.e., it is *impulse controllable* (or *controllable at infinity*), see [5, 12], and there exists a state feedback $u = Fx$ such that the closed loop system $\mathcal{E}\dot{x} = (\mathcal{A}+\mathcal{B}F)x$ is strangeness-free. Let us assume for simplicity that this preliminary feedback has already been performed, so that $\mathcal{E}\dot{x} = \mathcal{A}x$ is strangeness-free, i.e., the matrix

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}$$

is square and invertible. Numerical methods to compute such preliminary feedbacks are discussed in detail in [6, 5, 7].

From (2.5) we can read off that an initial condition is *consistent* if $A_2 x(t_0) + B_2 u(t_0) = 0$, and thus, the possible input functions u are restricted by this constraint. We then define the two subspaces

$$\mathbb{E}_d := \operatorname{Im}(\mathcal{E}^T), \quad \mathbb{E}_a := \ker(\mathcal{E}),$$

where Im and ker denote the *image* and *kernel*, respectively. Using the singular value decomposition, see [19],

$$U^T \mathcal{E} V = \left[\begin{array}{cc} \Sigma & 0 \\ 0 & 0 \end{array} \right],$$

with orthogonal U, V and nonsingular diagonal $\Sigma \in \mathbb{R}^{d,d}$ and the *Moore-Penrose inverse of* $\mathcal{E}$ given by

$$\mathcal{E}^+ = V^T \left[\begin{array}{cc} \Sigma^{-1} & 0 \\ 0 & 0 \end{array} \right] U \in \mathbb{R}^{n,m},$$

we can numerically compute four projectors

$$P_d := \mathcal{E}^+ \mathcal{E} = V^T \left[\begin{array}{cc} I_d & 0 \\ 0 & 0 \end{array} \right] V, \tag{2.6a}$$

$$P_a := I - \mathcal{E}^+ \mathcal{E} = V^T \left[\begin{array}{cc} 0 & 0 \\ 0 & I_{n-d} \end{array} \right] V, \tag{2.6b}$$

$$Q_d := \mathcal{E}\mathcal{E}^+ = U^T \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} U, \tag{2.6c}$$

$$Q_a := I - \mathcal{E}\mathcal{E}^+ = U^T \begin{bmatrix} 0 & 0 \\ 0 & I_{n-d} \end{bmatrix} U. \tag{2.6d}$$

Then P_d and P_a are orthogonal projectors onto the subspaces $\mathbb{E}_d$ and $\mathbb{E}_a$, respectively.

The computation of these projectors can be implemented in backward stable methods, the only subtlety is the usual decision which singular values are set to 0. The analysis for this numerical rank decision is well understood, see [16, 17]. Using the projectors, we can then partition the state additively as $x = x_d + x_a$ with the two parts

$$x_d := P_d x, \quad x_a := P_a x \tag{2.7}$$

and have the following result.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

THEOREM 2.2 ([3]). *Let the projectors P_d, P_a and Q_d, Q_a be defined as in (2.6) and the variables x_d and x_a as in (2.7). Then, $x = x_d + x_a$ is a solution of (2.2) if and only if x_d and x_a are solutions of the system*

$$\dot{x}_d = G_d x_d + B_d u, \tag{2.8a}$$
$$x_a = G_a x_d + B_a u, \tag{2.8b}$$

where

$$G_a := -(Q_a \mathcal{A} P_a)^+ (Q_a \mathcal{A} P_d), \quad G_d := \mathcal{E}^+ \mathcal{A}(P_d + G_a), \tag{2.9a}$$
$$B_a := -(Q_a \mathcal{A} P_a)^+ \mathcal{B}, \quad B_d := \mathcal{E}^+ \mathcal{B} + \mathcal{E}^+ \mathcal{A} B_a, \tag{2.9b}$$

respectively. Moreover, an initial value x_0 is consistent if and only if it satisfies

$$(P_a - G_a) x_0 = B_a u(t_0)$$

at the initial time t_0.

The solution set of system (2.8) can be used to compute the set of state reachable points of the first order system (2.2) from a consistent initial condition $x(t_0) = x_0$. This is discussed in the next section.

3. The reachable set in the first order case. Following the previous analysis, a reachable point x_1 of (2.2) can be determined by computing its differential component x_{d_1} and then the algebraic component x_{a_1} via the relations in (2.8). If x_0 is consistent, then $x_{d_0} = P_d x_0$ is an initial condition for the standard ordinary differential equation system $\dot{x}_d = G_d x_d + B_d u$ and the solution is

$$x_d = e^{G_d(t-t_0)} x_{d_0} + \int_{t_0}^{t} e^{G_d(t-\tau)} B_d u(\tau)\, d\tau. \tag{3.10}$$

The resulting algebraic component x_a is given by

$$x_a = G_a \left[e^{G_d(t-t_0)} x_{d_0} + \int_{t_0}^{t} e^{G_d(t-\tau)} B_d u(\tau)\, d\tau \right] + B_a u. \tag{3.11}$$

This shows that the state responses x_d and x_a are uniquely determined by the initial condition x_0, the control input $u(\tau)$ for $t_0 \leq \tau \leq t$ and the initial time t_0. If the initial time is $t_0 = 0$ and the initial condition is $x_0 = 0$, then we have $x_d(0) = P_d x_0 = 0$ and hence, the solution (3.10) is given by

$$x_d = \int_0^t e^{G_d(t-\tau)} B_d u(\tau)\, d\tau. \tag{3.12}$$

With the operator

$$\mathcal{L}_d(u,t) := \int_0^t e^{G_d(t-\tau)} B_d u(\tau)\, d\tau,$$

the set of reachable points x_{d_1} from $x(0) = 0$ is the image space of $\mathcal{L}_d(u,t)$ which, see [13], is the image space of the symmetric positive semidefinite *Gramian* matrix

$$W(p,t) := \int_0^t p(\tau)^2 e^{G_d(t-\tau)} B_d B_d^T e^{G_d^T(t-\tau)}\, d\tau, \tag{3.13}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

where $p(\tau)$ is a polynomial which is not identically zero. Moreover, by defining the subspace

$$\mathcal{X}_d := \mathrm{Im}\,(\mathcal{C}_d)\,, \tag{3.14}$$

where $\mathcal{C}_d = \begin{bmatrix} B_d & G_d B_d & \cdots & G_d^{n-1} B_d \end{bmatrix}$ is the *controllability matrix* of the dynamical part, it follows as in [12] that

$$\mathrm{Im}[W(p,t)] = \mathcal{X}_d. \tag{3.15}$$

The algebraic part $x_a = G_a x_d + B_a u$ imposes restrictions in the reachability set. Thus, to compute a reachable point x_1, assume that x_{d_1} is a reachable point due to the state response x_d for some input u_1 at a finite time $t_1 > 0$. Then at t_1, the algebraic component x_{a_1} is given by $x_{a_1} = G_a x_{d_1} + B_a u_1(t_1)$. Since $x_{d_1} \in \mathcal{X}_d$, we can write $x_{d_1} = \mathcal{C}_d w$ for some $w \in \mathbb{R}^{nm}$. Hence, x_{a_1} takes the form

$$x_{a_1} = G_a \mathcal{C}_d w + B_a c,$$

where $c = u_1(t_1) \in \mathbb{R}^m$. Defining the set

$$\begin{aligned} \mathcal{X}_a :=& \{x_{a_1} \in \mathbb{R}^n \mid x_{a_1} = G_a \mathcal{C}_d w + B_a c,\ c = u(t_1)\} \\ =& \mathrm{Im}[G_a \mathcal{C}_d \quad B_a], \end{aligned} \tag{3.16}$$

we have the complete characterization of the set $\mathcal{R}_0$ of reachable points of system (2.2) from an initial condition $x_0 = 0$ at $t_0 = 0$, given by

$$\mathcal{R}_0 = \mathcal{X}_d + \mathcal{X}_a = \{x_{d_1} + x_{a_1} \mid x_{d_1} \in \mathcal{X}_d,\ x_{a_1} \in \mathcal{X}_a\}\,; \tag{3.17}$$

see [13, 14] for details. Since $\mathcal{X}_d = \mathrm{Im}(\mathcal{C}_d)$ and $\mathcal{X}_a = \mathrm{Im}[G_a \mathcal{C}_d \quad B_a]$, the set $\mathcal{R}_0$ can be computed in a numerically backward stable way using the matrices defined in (2.9). Moreover, $\dim(\mathcal{R}_0) = \dim(\mathcal{X}_d) + \dim(\mathcal{X}_a)$, where dim refers to the dimension of a subspace.

When we start from an arbitrary consistent initial condition $x(t_0) = x_0$, then by performing a time shift and considering $x - x_0$ we obtain as reachable set the affine space $x_0 + \mathcal{R}_0$. A first order example illustrating the computational process is presented in [13, 14]. In the following section we use these results to compute the set of reachable points for a second order descriptor system in trimmed first order form.

4. Reachability set for second order descriptor systems. To extend the procedure to compute the reachable set to second order descriptor systems of the form (1.1), we first reformulate the strangeness-free second order system by a derivative array approach directly from the behavior form of the second order system, see [29, 35],

$$\mathbf{M}\ddot{z}(t) + \mathbf{D}\dot{z}(t) = \mathbf{K}z(t), \tag{4.18}$$

where $\mathbf{M} := [M \quad 0] \in \mathbb{R}^{n,n+m}$, $\mathbf{D} := [D \quad 0] \in \mathbb{R}^{n,n+m}$, $\mathbf{K} := [-K \quad B] \in \mathbb{R}^{n,n+m}$, by introducing the new variable $z = \begin{bmatrix} x^T & u^T \end{bmatrix}^T$. Then, the derivative array takes the form

$$\mathcal{M}_\mu \ddot{z}_\mu + \mathcal{L}_\mu \dot{z}_\mu = \mathcal{N}_\mu z_\mu, \tag{4.19}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

where $z_\mu = \begin{bmatrix} z & \dot{z} & \ddot{z} & \cdots & z^{(\mu)} \end{bmatrix}^T$,

$$\mathcal{M}_\mu = \begin{bmatrix} \mathbf{M} & 0 & 0 & \cdots & 0 \\ \mathbf{D} & \mathbf{M} & 0 & \cdots & 0 \\ -\mathbf{K} & \mathbf{D} & \mathbf{M} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -\mathbf{K} & \mathbf{D} & \mathbf{M} \end{bmatrix}, \quad \mathcal{L}_\mu = \begin{bmatrix} \mathbf{D} & 0 & \cdots & 0 \\ -\mathbf{K} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \mathcal{N}_\mu = \begin{bmatrix} \mathbf{K} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The strangeness-free system is obtained by computing projection matrices Z_0, Z_1, Z_2 and Z_3, see [35], such that

$$\begin{bmatrix} \widehat{M}_1 \\ 0 \\ 0 \end{bmatrix} \ddot{z} + \begin{bmatrix} \widehat{D}_1 \\ \widehat{D}_2 \\ 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \widehat{K}_1 \\ \widehat{K}_2 \\ \widehat{K}_3 \end{bmatrix} z, \tag{4.20}$$

where

$$\widehat{M}_1 = Z_0^T \mathbf{M} \in \mathbb{R}^{d_1, n+m}, \quad \widehat{D}_1 = Z_0^T \mathbf{D} \in \mathbb{R}^{d_1, n+m},$$
$$\widehat{K}_1 = Z_0^T \mathbf{K} \in \mathbb{R}^{d_1, n+m}, \quad \widehat{D}_2 = Z_1^T Z_2^T \mathcal{L}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \in \mathbb{R}^{d_2, n+m},$$
$$\widehat{K}_2 = Z_1^T Z_2^T \mathcal{N}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \in \mathbb{R}^{d_2, n+m},$$
$$\widehat{K}_3 = Z_3^T \mathcal{N}_\mu \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} \in \mathbb{R}^{a, n+m}.$$

In the strangeness-free system (4.20), the matrices $\widehat{M}_1$, $\widehat{D}_2$ and $\widehat{K}_3$ are of full row rank and, furthermore by our assumption that the system is regular, also the matrix

$$\widehat{H} := \begin{bmatrix} \widehat{M}_1 \\ \widehat{D}_2 \\ \widehat{K}_3 \end{bmatrix} \in \mathbb{R}^{n, n+m} \tag{4.21}$$

is of full row rank n. Moreover, (4.20) can be represented (in the original variables) as

$$M_1 \ddot{x} + D_1 \dot{x} + K_1 x = B_1 u, \tag{4.22a}$$
$$D_2 \dot{x} + K_2 x = B_2 u, \tag{4.22b}$$
$$K_3 x = B_3 u, \tag{4.22c}$$

where M_i, D_i and K_i, B_i are obtained from $\widehat{M}_i$, $\widehat{D}_i$ and $\widehat{K}_i$, respectively, by inserting the partitioned vector z and taking all state terms to the left.

As in the first order case, system (4.22) may, however, not be strangeness-free as a free system with $u = 0$, since the matrix

$$H := \begin{bmatrix} M_1 \\ D_2 \\ K_3 \end{bmatrix} \in \mathbb{R}^{n,n}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

may not be of full row rank n. But since the behavior system is strangeness-free, again there exists a feedback $u = -G\dot{x} - Fx + \tilde{u}$ such that in the closed loop system

$$M_1\ddot{x} + (D_1 + B_1 G)\dot{x} + (K_1 + B_1 F)x = B_1\tilde{u}, \tag{4.23a}$$

$$(D_2 + B_2 G)\dot{x} + (K_2 + B_2 F)x = B_2\tilde{u}, \tag{4.23b}$$

$$(K_3 + B_3 F)x = B_3\tilde{u}, \tag{4.23c}$$

the matrix

$$H_F := \begin{bmatrix} M_1 \\ D_2 + B_2 G \\ K_3 + B_3 F \end{bmatrix} \in \mathbb{R}^{n,n}$$

is nonsingular; see [24].

In order to apply the results from the first order case, we transform system (4.23) into a trimmed first-order formulation that does not need unnecessary smoothness requirements. For this, we determine an orthogonal matrix W (this can be done in a numerically stable way via a singular value or QR decomposition with pivoting [19]) such that in

$$H_F W = \begin{bmatrix} \tilde{M}_{11} & 0 & 0 \\ \tilde{D}_{21} & \tilde{D}_{22} & 0 \\ \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} \end{bmatrix} \tag{4.24}$$

the matrices $\tilde{M}_{11} \in \mathbb{R}^{d_1,d_1}$, $\tilde{D}_{22} \in \mathbb{R}^{d_2,d_2}$, $\tilde{K}_{33} \in \mathbb{R}^{a,a}$ are square and nonsingular.

Then, by performing a change of basis, i.e., by setting $x = Ww$, we obtain $M_1 W = [\tilde{M}_{11}\, 0\, 0]$, $(D_1 + B_1 G)W = [\tilde{D}_{11}\, \tilde{D}_{12}\, \tilde{D}_{13}]$, $(D_2 + B_2 G)W = [\tilde{D}_{21}\, \tilde{D}_{22}\, 0]$ and $(K_j + B_j F)W = [\tilde{K}_{j1}\, \tilde{K}_{j2}\, \tilde{K}_{j3}]$, $j = 1, 2, 3$. In this setting, system (4.23) can be represented as

$$\begin{bmatrix} \tilde{M}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \\ \ddot{w}_3 \end{bmatrix} + \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ \tilde{D}_{21} & \tilde{D}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} + \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \\ \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \tilde{u}.$$

By introducing a new variable $v = \dot{w}_1$, we obtain the strangeness-free first order system

$$\begin{bmatrix} \tilde{M}_{11} & \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ 0 & \tilde{D}_{21} & \tilde{D}_{22} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_{d_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} + \begin{bmatrix} 0 & \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} \\ 0 & \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \\ 0 & \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} \\ -I_{d_1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix} \tilde{u}. \tag{4.25}$$

Note that v can be interpreted as the *velocity vector* of the dynamic part.

This formulation is nice but it is not expressed in the original variables, so that the physical meaning of the variables has changed. To avoid this difficulty, by inserting $w = W^T x$ in terms of the original variables, one obtains

$$\begin{bmatrix} \tilde{M}_{11} & (D_1 + B_1 G) \\ 0 & (D_2 + B_2 G) \\ 0 & 0 \\ 0 & W_1^T \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 & K_1 + B_1 F \\ 0 & K_2 + B_2 F \\ 0 & K_3 + B_3 F \\ -I_{d_1} & 0 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix} \tilde{u},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

where $W_1^T = [I_{d_1}\, 0\, 0]W^T$. Reordering the block rows we obtain

$$\begin{bmatrix} \tilde{M}_{11} & (D_1 + B_1 G) \\ 0 & (D_2 + B_2 G) \\ 0 & W_1^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 & K_1 + B_1 F \\ 0 & K_2 + B_2 F \\ -I_{d_1} & 0 \\ 0 & K_3 + B_3 F \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ B_3 \end{bmatrix} \tilde{u}.$$

This system is strangeness-free as a free system with $\tilde{u} = 0$, and we write this as

$$L\dot{\xi} = S\xi + \tilde{B}\tilde{u}, \tag{4.26}$$

where

$$L = \begin{bmatrix} \tilde{M}_{11} & (D_1 + B_1 G) \\ 0 & (D_2 + B_2 G) \\ 0 & W_1^T \\ 0 & 0 \end{bmatrix}, \quad S = -\begin{bmatrix} 0 & K_1 + B_1 F \\ 0 & K_2 + B_2 F \\ -I_{d_1} & 0 \\ 0 & K_3 + B_3 F \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ B_3 \end{bmatrix}, \quad \xi = \begin{bmatrix} v \\ x \end{bmatrix}, \quad v = W_1^T \dot{x}.$$

Since $v(t) = W_1^T \dot{x}(t)$, it is clear that the components of a reachable velocity vector $v(t)$ will lie on hyperplanes defined by $W_1^T \dot{x}(t)$ for $t \geq 0$. Recall that in Example 1.3 we have discussed that the velocity components of a constrained mechanical system are restricted in some directions and as a result the velocity reachable points are also. The velocity directions that one can reach are defined by the relation $v(t) = W_1^T \dot{x}(t)$ and the other components of the velocity vector $v \in \mathbb{R}^n$ will remain unchanged. After this restriction of the velocity vector, we can apply the results of the first order case and obtain

$$\dot{\xi}_d(t) = G_d \xi_d(t) + B_d \tilde{u}(t), \tag{4.27a}$$

$$\xi_a(t) = G_a \xi_d(t) + B_a \tilde{u}(t), \tag{4.27b}$$

where

$$G_a := -(Q_a S P_a)^+ Q_a S P_d, \quad G_d := L^+ S(P_d + G_a), \tag{4.28a}$$

$$B_a := -(Q_a S P_a)^+ \tilde{B}, \quad B_d := L^+ \tilde{B} + L^+ S B_a, \tag{4.28b}$$

with projectors

$$P_d = L^+ L, \quad P_a = I - L^+ L, \quad Q_d = LL^+ = \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_a = I - LL^+. \tag{4.29}$$

In (4.27), the extended state vector ξ satisfies $\xi = \xi_d + \xi_a$ where $\xi_d = \begin{bmatrix} v_d^T, & x_d^T \end{bmatrix}^T$ and $\xi_a = \begin{bmatrix} v_a^T, & x_a^T \end{bmatrix}^T$. It follows that an initial value

$$\xi_0 = \begin{bmatrix} W_1^T \dot{x}(t_0) \\ x(t_0) \end{bmatrix}$$

is consistent if and only if it satisfies

$$(P_a - G_a)\xi_0 = B_a \tilde{u}(t_0)$$

at time t_0. Hence, the reachable points for a given control input $\tilde{u}$ can be determined from the relations in (4.27).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

Let us first discuss the question of computing the reachable position (for the dynamic part) and the reachable velocity points from zero initial conditions, i.e., $x(0) = x_0 = 0$ and $v(0) = v_0 = 0$. Corresponding to these initial conditions the solution of (4.27a) is given by

$$\xi_d = \int_0^t e^{G_d(t-\tau)} B_d \tilde{u}(\tau) d\tau,$$

and hence,

$$v_d = \begin{bmatrix} I_{d_1} & 0 \end{bmatrix} \left(\int_0^t e^{G_d(t-\tau)} B_d \tilde{u}(\tau) d\tau \right), \tag{4.30a}$$

$$x_d = \begin{bmatrix} 0 & I_n \end{bmatrix} \left(\int_0^t e^{G_d(t-\tau)} B_d \tilde{u}(\tau) d\tau \right). \tag{4.30b}$$

If we partition the matrix S as

$$S = \left[\begin{array}{c|ccc} 0 & K_{11} & K_{12} & K_{13} \\ \hline 0 & K_{21} & K_{22} & K_{23} \\ I_{d_1} & 0 & 0 & 0 \\ 0 & K_{31} & K_{32} & K_{33} \end{array}\right],$$

and partition the projectors Q_a and P_a accordingly, then we obtain the following consistency conditions from the algebraic relation (4.27b):

$$v_a = 0, \tag{4.31a}$$

$$x_a = \begin{bmatrix} 0 & I_n \end{bmatrix} \left[G_a \left(\int_0^t e^{G_d(t-\tau)} B_d \tilde{u}(\tau) d\tau \right) + B_a \tilde{u} \right]. \tag{4.31b}$$

Hence, the reachable position and velocity points are determined by (4.30) and (4.31). Furthermore, since $v_a = 0$, the reachable vectors $v = v_d$ are determined by (4.30a). We thus have proved the following reachability conditions.

Theorem 4.1. *Consider a second order system in the regularized first order form (4.25).*

1. *A state vector $x_1 := x_{d_1} + x_{a_1}$ is* reachable from a consistent initial condition $x(t_0) = x_0$ *if there exists a control input $\tilde{u}(t)$, and a finite time $t_1 > 0$ such that $x_d(t_1) = x_{d_1}$ and $x_a(t_1) = x_{a_1}$.*
2. *An extended state vector $\xi_1 := \xi_{d_1} + \xi_{a_1}$ is* reachable from a consistent initial condition $\xi(t_0) = \xi_0$ *if there exists a control input $\tilde{u}(t)$ and a finite time $t_1 > 0$ such that $\xi(t_1) = \xi_1$.*

We can also give explicit formulas for the reachability sets and the associated controls. The reachable set $\mathcal{R}_{\xi_0}$ associated with the trimmed first order system (4.25) is given by

$$\mathcal{R}_{\xi_0} = \mathcal{W}_d + \mathcal{W}_a, \tag{4.32}$$

where

$$\begin{aligned} \mathcal{W}_d &:= \mathrm{Im}\,(\mathcal{C}_d)\,, \quad \text{with } \mathcal{C}_d = \begin{bmatrix} B_d & G_d B_d & \cdots & G_d^{n-1} B_d \end{bmatrix}, \\ \mathcal{W}_a &:= \mathrm{Im}\left(\begin{bmatrix} G_a \mathcal{C}_d & B_a \end{bmatrix}\right). \end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

Hence, the set of reachable extended state vectors ξ_1 is contained in the subspace $\mathcal{R}_{\xi_0}$. Since the system (4.26) is strangeness-free, the control input is given by

$$\tilde{u} = B_d^T e^{G_d^T(t-\tau)} k, \quad 0 \le \tau \le t, \tag{4.33}$$

where $k \in \mathbb{R}^{n+d_1}$ satisfies $W(t)k = \xi_{d_1}$ ($W(t)$ is the associated Gramian matrix). With the subspaces

$$\begin{aligned} \mathcal{X}_d &:= \operatorname{Im}\left([0 \quad I_n]\, \mathcal{C}_d\right), \\ \mathcal{X}_a &:= \operatorname{Im}\left([0 \quad I_n] \begin{bmatrix} G_a\mathcal{C}_d & B_a \end{bmatrix}\right), \end{aligned}$$

the set of reachable position vectors x_1 is contained in the subspace

$$\mathcal{R}_{x_0} = \mathcal{X}_d + \mathcal{X}_a. \tag{4.34}$$

Note that the derivative array approach and thus also the construction of the reachable sets can be extended in a similar way for systems of order higher than two.

Remark 4.2. We have discussed at several places that the strangeness-free behavior model may not be strangeness-free for $u(t) = 0$. This situation can be circumvented by using feedback controls of the form $u(t) = Fx(t) + \tilde{u}(t)$ and $u(t) = -G\dot{x}(t) + Fx(t) + \tilde{u}(t)$ for first and second order descriptor systems, respectively. Recall that while computing the reachable space for the first order descriptor system we have used the strangeness-free behavior model and the control input $u(t)$ instead of $\tilde{u}(t)$. Hence, we have considered a smooth control input $u(t)$. On the other hand, by using a preliminary feedback control, we do not need smoothness conditions on the input function $\tilde{u}(t)$. This situation is demonstrated in the second order descriptor system, where the control input $\tilde{u}(t)$ takes the form $\tilde{u}(t) = B_d^T e^{G_d^T(t-\tau)} k,\ 0 \le \tau \le t$ for first as well as second order descriptor systems.

5. Numerical procedure and example. We have demonstrated how to obtain the reachability sets of second order systems of the form (1.1) in a numerically stable way in the previous section. We have implemented this construction in the following numerically backward stable procedure. The first step is a procedure for the construction of the strangeness-free form (4.20), adapted from [35]. From this form we can then explicitly construct the reachability sets.

Algorithm 5.1. *Construction of regularized second order descriptor system.*

Input: *Matrices* $\mathbf{M} := [M \quad 0]$, $\mathbf{D} := [D \quad 0]$ *and* $\mathbf{K} := [-K \quad B]$ *constructed from the second order descriptor system (1.1).*

Output: *The projection matrices* Z_0, Z_1, Z_2 *and* Z_3 *that are required to obtain the strangeness-free system* (4.20).

Steps:

1. *Start with* $\mu = 0$.
2. *Form the derivative array according to* (4.19).
3. *Compute 'a' from the relation: rank*$([\mathcal{M}_\mu \quad \mathcal{L}_\mu]) = (\mu+1)n - a$.
4. *Compute* Z *such that* $Z^T\mathcal{M}_\mu = 0$ *and* Z_3 *such that* $Z_3^T[\mathcal{M}_\mu \quad \mathcal{L}_\mu] = 0$.
5. *Let* R_z *be a basis matrix for the range space of* Z^TZ_3. *Then, compute* $Z_2 = ZR_z$.
6. *Compute* T_3 *such that* $Z_3^T\mathcal{N}_\mu[I \quad 0 \quad \cdots \quad 0]^T T_3 = 0$.
7. *Compute the rank of* $Z_2^T\mathcal{L}_\mu[I \quad 0 \quad \cdots \quad 0]^T T_3$ *and denote it as* d_2.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

8. *Compute the singular value decomposition (SVD) of* $Z_2^T \mathcal{L}_\mu [I \quad 0 \quad \cdots \quad 0]^T$ *and write it as*

$$Z_2^T \mathcal{L}_\mu [I \quad 0 \quad \cdots \quad 0]^T = U_{z_2} S_{z_2} V_{z_2}^T.$$

Then, form Z_1 *by taking the first* d_2 *columns of* U_{z_2}.

9. *Compute* T_2 *such that* $Z_1^T Z_2^T \mathcal{L}_\mu [I \quad 0 \quad \cdots \quad 0]^T T_3 T_2 = 0$.
10. *Compute the SVD of* $\mathbf{M} T_3 T_2 = U_m S_m V_m^T$. *Denote* d_1 *as the rank of* S_m. *Then form* Z_0 *by taking the first* d_1 *columns of* U_m.
11. *Check if* $a + d_2 + d_1 = n$ *then* **stop**, *otherwise increase the value of* μ *by one and follow the procedure from Step 2.*

The only critical point in this procedure is the decision about the numerical rank. However, as has been shown in [26], and improved in [21], if there is doubt about the rank decision in the construction of the strangeness-free form one just goes one level further to obtain a better rank decision.

Algorithm 5.2. *Computation of reachable sets.*

Input: *The matrices* M_1, D_1, D_2, K_i *and* B_i *for* $i = 1, 2, 3$, *associated with the strangeness-free system* (4.22), *the feedback gain matrices* F *and* G *to obtain* (4.23).

Output: *The matrices* W_d, W_a, X_d *and* X_a *that are required to compute the reachable subspaces.*

Steps:

1. *Construct the matrix* $H_F := \begin{bmatrix} M_1 \\ D_2 + B_2 G \\ K_3 + B_3 F \end{bmatrix}$.
2. *Compute the orthogonal matrix* W *(by performing either singular value or QR decomposition with pivoting) such that* (4.24) *holds.*
3. *Compute the matrices* L, S, $\tilde{B}$ *as in* (4.26), *and the projectors* P_d, P_a, Q_d, Q_a *as in* (4.29).
4. *Compute the matrices* G_d, G_a, B_d *and* B_a *as in* (4.28).
5. *Compute* $\mathcal{C}_d = \begin{bmatrix} B_d & G_d B_d & \cdots & G_d^{n-1} B_d \end{bmatrix}$ *and* $\mathcal{C}_a := \begin{bmatrix} G_a \mathcal{C}_d & B_a \end{bmatrix}$.
6. *Compute the SVD of* $\mathcal{C}_d$ *and write it as* $\mathcal{C}_d = U_{cd} S_{cd} V_{cd}^T$. *Then construct a matrix* W_d *by taking the columns of* U_{cd} *corresponding to the non-zero diagonal elements of* S_{cd}. *Similarly, write* $\mathcal{C}_a = U_{ca} S_{ca} V_{ca}^T$ *by computing the SVD of* $\mathcal{C}_a$, *and then construct a matrix* W_a *by taking the columns of* U_{ca} *corresponding to the non-zero diagonal elements of* S_{ca}.
7. *Compute the SVD of* $[0 \quad I_n]\, \mathcal{C}_d$ *and write it as* $[0 \quad I_n]\, \mathcal{C}_d = U_{xd} S_{xd} V_{xd}^T$. *Then construct a matrix* X_d *by taking the columns of* U_{xd} *corresponding to the non-zero diagonal elements of* S_{xd}. *Similarly, write* $[0 \quad I_n]\, \mathcal{C}_a = U_{xa} S_{xa} V_{xa}^T$ *by computing the SVD of* $[0 \quad I_n]\, \mathcal{C}_a$, *and then construct a matrix* X_a *by taking the columns of* U_{xa} *corresponding to the non-zero diagonal elements of* S_{xa}.

The subspaces $\mathcal{W}_d$ and $\mathcal{W}_a$ are spanned by the columns of the matrices W_d and W_a, respectively. Similarly, $\mathcal{X}_d$ and $\mathcal{X}_a$ are spanned by the columns of X_d and X_a, respectively. Then the reachable subspaces are as in (4.32) and (4.34).

To illustrate the technique for the computation of the reachable set, we consider the model of the three-link mobile manipulator from [20], see Example 1.3, for which the concrete data can be found in [14]. Carrying out the regularization procedure, it follows that this system has strangeness-index $\mu = 2$ in the

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

behavior setting, and we obtain a strangeness-free behavior model (4.20), where

$$
\begin{aligned}
\widehat{M}_1 &= \begin{bmatrix} -14.5135 & 42.3643 & -38.6090 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\\
\widehat{D}_1 &= \begin{bmatrix} 4.7430 & 4.8373 & -4.8373 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\\
\widehat{K}_1 &= \begin{bmatrix} -83.9703 & 10.6511 & -54.9365 & -0.1875 & -0.6330 & 0.6752 & -1.1064 & -0.2019 \end{bmatrix},\\
\widehat{D}_2 &= 10^{-13}\begin{bmatrix} -0.2297 & 0.0275 & -0.7357 & -0.0000 & -0.0090 & 0.0016 & -0.0029 & -0.0077 \end{bmatrix},\\
\widehat{K}_2 &= \begin{bmatrix} 0.6477 & 0.2356 & -0.8514 & -0.0034 & 0.0048 & -0.0000 & 0.0025 & 0.0023 \end{bmatrix},\\
\widehat{K}_3 &= \begin{bmatrix} 0.3913 & 0.2676 & -0.8869 & -0.0038 & 0.0054 & -0.0000 & 0.0028 & 0.0026\\ 4.2024 & 2.3386 & 1.0103 & -0.0334 & 0.0473 & -0.0000 & 0.0246 & 0.0227\\ 4.8339 & 3.7396 & 1.6016 & -0.0534 & 0.0757 & -0.0000 & 0.0394 & 0.0363 \end{bmatrix}.
\end{aligned}
$$

The matrix $\widehat{H}$ in (4.21) is of full row rank 5, when setting the tolerance equal to the machine precision *eps* in computing the rank. The resulting system is also strangeness-free for $u = 0$, since the rank of H is 5. However, since the matrix H is nearly singular, we may improve the robustness of the system by using a preliminary feedback control $u = -G\dot{x} - Fx + \tilde{u}$, with

$$
G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 5 & 0 & 0\\ 0 & 0 & 0 & 10 & 0 \end{bmatrix}, \quad F = 0,
$$

which leads to an H_F which is robustly of full row rank.

To obtain a first order formulation we choose the following orthogonal matrix

$$
W = \begin{bmatrix} -0.2455 & 0.0804 & 0.7585 & 0.5982 & -0.0100\\ 0.7165 & -0.2347 & -0.2289 & 0.6158 & -0.0026\\ -0.6530 & -0.2877 & -0.5363 & 0.4508 & 0.0009\\ 0 & -0.9250 & 0.2908 & -0.2444 & -0.0005\\ 0 & 0.0000 & 0.0076 & 0.0071 & 0.9999 \end{bmatrix}
$$

which is obtained by computing a QR decomposition of H_F^T. Then, following the discussion in Section 4, we obtain an associated first order model $L\dot{\xi} = S\xi + \tilde{B}u$, where

$$
L = \begin{bmatrix} 59.1273 & 4.7430 & 4.8373 & -10.3692 & -2.0190 & 0\\ 0 & -0.0000 & 0.0000 & 0.0124 & 0.0229 & -0.0000\\ 0 & -0.2455 & 0.7165 & -0.6530 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
S = \begin{bmatrix} 0 & -83.9703 & 10.6511 & -54.9365 & -0.1875 & -0.6330\\ 0 & 0.6477 & 0.2356 & -0.8514 & -0.0034 & 0.0048\\ 1.0000 & 0 & 0 & 0 & 0 & 0\\ 0 & 0.3913 & 0.2676 & -0.8869 & -0.0038 & 0.0054\\ 0 & 4.2024 & 2.3386 & 1.0103 & -0.0334 & 0.0473\\ 0 & 4.8339 & 3.7396 & 1.6016 & -0.0534 & 0.0757 \end{bmatrix},
$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

$$\tilde{B} = \begin{bmatrix} 0.6752 & -1.1064 & -0.2019 \\ -0.0000 & 0.0025 & 0.0023 \\ 0 & 0 & 0 \\ -0.0000 & 0.0028 & 0.0026 \\ -0.0000 & 0.0246 & 0.0227 \\ -0.0000 & 0.0394 & 0.0363 \end{bmatrix}.$$

The subspaces $\mathcal{W}_d$ and $\mathcal{W}_a$ corresponding to the trimmed first order formulation of the second order system are spanned by the columns of

$$W_d = \begin{bmatrix} -0.6642 & 0.7365 \\ -0.2664 & -0.0903 \\ 0.5678 & 0.4962 \\ -0.3576 & -0.3963 \\ 0.1939 & 0.2149 \\ -0.0000 & -0.0000 \end{bmatrix}, \quad W_a = \begin{bmatrix} -0.0001 & 0.0145 & 0.1274 \\ -0.0041 & 0.3219 & -0.9040 \\ -0.0078 & 0.6566 & 0.0169 \\ -0.0070 & 0.5994 & 0.3584 \\ 0.0038 & -0.3250 & -0.1943 \\ 0.9999 & 0.0119 & -0.0002 \end{bmatrix}$$

and the subspaces $\mathcal{X}_d$ and $\mathcal{X}_a$ are spanned by the columns of

$$X_d = \begin{bmatrix} -0.2953 & -0.8571 \\ 0.7572 & 0.0594 \\ -0.5122 & 0.4498 \\ 0.2777 & -0.2439 \\ -0.0000 & 0.0000 \end{bmatrix}, \quad X_a = \begin{bmatrix} -0.0041 & -0.3223 & 0.9107 \\ -0.0078 & -0.6566 & -0.0186 \\ -0.0070 & -0.5993 & -0.3627 \\ 0.0038 & 0.3250 & 0.1967 \\ 0.9999 & -0.0119 & 0.0002 \end{bmatrix},$$

respectively. Hence, the reachable spaces are $\mathcal{R}_{\xi_0} = \mathcal{W}_d + \mathcal{W}_a$ and $\mathcal{R}_{x_0} = \mathcal{X}_d + \mathcal{X}_a$.

6. Conclusion. We have presented a procedure to compute the reachable set for second order linear time-invariant descriptor systems. We have shown how to obtain a strangeness-free behavior model corresponding to the original model via a derivative array approach. The strangeness-free system is then decoupled into a differential part and an algebraic part with the help of a projection method. The coefficient matrices of the projected system are used to define two subspaces and the addition of these two subspaces gives the reachable space, from the origin, of the original descriptor system. To apply this technique for second order systems, the constructed strangeness-free behavior model is transformed into a first order system by a trimmed first order formulation, which is then used to compute the reachable vectors.

REFERENCES

[1] P. Antognetti and G. Massobrio. *Semiconductor Device Modeling with SPICE.* McGraw-Hill, New York, 1997.

[2] M. Athans and P.L. Falb. *Optimal Control.* McGraw-Hill, New York, 1966.

[3] A.K. Baum and V. Mehrmann. Positivity characterization of nonlinear DAEs, part I: Decomposition of differential and algebraic equations using projections. Preprint-18, Institut für Mathematik, TU Berlin, Berlin, 2013.

[4] W. Blajer and K. Kołodziejczyk. Control of under actuated mechanical systems with servo-constraints. *Nonlinear Dynam.*, 50:781–791, 2007.

[5] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N.K. Nichols. Feedback design for regularizing descriptor systems. *Linear Algebra Appl.*, 299:119–151, 1999.

[6] A. Bunse-Gerstner, V. Mehrmann, and N.K. Nichols. Regularization of descriptor systems by derivative and proportional state feedback. *SIAM J. Matrix Anal. Appl.*, 13:46–67, 1992.

[7] A. Bunse-Gerstner, V. Mehrmann, and N.K. Nichols. Regularization of descriptor systems by output feedback. *IEEE Trans. Automat. Control*, 39:1742–1748, 1994.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 301-316, August 2017.
http://repository.uwyo.edu/ela

[8] R. Byers, V. Mehrmann, and H. Xu. Trimmed linearizations for structured matrix polynomials. *Linear Algebra Appl.*, 429:2373–2400, 2008.

[9] S.L. Campbell. A general form for solvable linear time varying singular systems of differential equations. *SIAM J. Math. Anal.*, 18:1101–1115, 1997.

[10] S.L. Campbell, P. Kunkel, and V. Mehrmann. *Regularization of Linear and Nonlinear Descriptor Systems.* SIAM, Philadelphia, PA, 17–34, 2012.

[11] D. Cobb. Controllability, observability and duality in singular systems. *IEEE Trans. Automat. Control*, 29:1076–1082, 1984.

[12] L. Dai. *Singular Control Systems.* Springer-Verlag, Berlin, 1989.

[13] S. Datta and V. Mehrmann. Computation of state reachable points of descriptor systems. *Proceedings of the 53rd IEEE Conference on Decision and Control* (Los Angeles, California, December 15–17, 2014), 2014.

[14] S. Datta and V. Mehrmann. Computation of state reachable points of linear time invariant descriptor systems. Preprint-17, Institut für Mathematik, TU Berlin, Berlin, 2014.

[15] J.W. Demmel and B. Kågström. Computing stable eigendecompositions of matrix pencils. *Linear Algebra Appl.*, 88:139–186, 1987.

[16] J.W. Demmel and B. Kågström. The generalized Schur decomposition of an arbitrary pencil $\lambda A - B$, Part I. *ACM Trans. Math. Software*, 19:160–174, 1993.

[17] J.W. Demmel and B. Kågström. The generalized Schur decomposition of an arbitrary pencil $\lambda A - B$, Part II. *ACM Trans. Math. Software*, 19:185–201, 1993.

[18] Dynasim AB. *Dymola, Multi-Engineering Modelling and Simulation.* Ideon Research Park, SE-223 70 Lund, Sweden, 2006.

[19] G.H. Golub and C.F. Van Loan. *Matrix Computations*, third edition. The Johns Hopkins University Press, Baltimore, 1996.

[20] M. Hou and P.C. Müller. LQ and tracking control of descriptor systems with application to constrained manipulator. Technical Report, Sicherheitstechnische Regelungs-und Meßtechnik, Universität Wuppertal, Gauß-Straße 20, D-5600 Wuppertal 1, Germany, 1994.

[21] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution.* EMS Publishing House, Zürich, 2006.

[22] P. Kunkel, V. Mehrmann, and W. Rath. Analysis and numerical solution of control problems in descriptor form. *Math. Control Signals Systems*, 14:29–61, 2001.

[23] R. Lamour, R. März, and C. Tischendorf. *Differential-Algebraic Equations: A Projector Based Analysis.* Springer-Verlag, Berlin, 2013.

[24] P. Losse and V. Mehrmann. Controllability and observability of second order descriptor systems. *SIAM J. Control Optim.*, 47:1351–1379, 2008.

[25] The MathWorks, Inc. *Simscape.* Natick, MA, 2013.

[26] R.M.M. Mattheij and P.M.E.J. Wijckmans. Conditioning of two deck differential algebraic equations. Technical Report RANA 97-19, Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven, The Netherlands, 1997.

[27] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem.* Springer-Verlag, Berlin, 1991.

[28] V. Mehrmann. *Index Concepts for Differential-Algebraic Equations.* Springer-Verlag, Heidelberg, 2015.

[29] V. Mehrmann and C. Shi. Transformation of high order linear differential-algebraic systems to first order. *Numer. Algorithms*, 42:281–307, 2006.

[30] B. Pal and B. Chaudhuri. *Robust Control in Power Systems.* Springer, New York, 2005.

[31] J.W. Polderman and J.C. Willems. *Introduction to Mathematical Systems Theory: A Behavioural Approach.* Springer-Verlag, New York, 1998.

[32] P.J. Rabier and W.C. Rheinboldt. Classical and generalized solutions of time-dependent linear differential-algebraic equations. *Linear Algebra Appl.*, 245:259–293, 1996.

[33] P.J. Rabier and W.C. Rheinboldt. *Nonholonomic Motion of Rigid Mechanical Systems from a DAE Viewpoint.* SIAM Publications, Philadelphia, 2000.

[34] R. Riaza. *Differential-Algebraic Systems. Analytical Aspects and Circuit Applications.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ., 2008.

[35] L. Scholz. A derivative array approach for linear second order differential-algebraic systems. *Electron. J. Linear Algebra*, 22:310–347, 2011.

[36] P. Van Dooren. The computation of Kronecker's canonical form of a singular pencil. *Linear Algebra Appl.*, 27:103–141, 1979.

[37] G.C. Verghese, B.C. Lévy, and T. Kailath. A general state space for singular systems. *IEEE Trans. Automat. Control*, AC-26:811–831, 1981.