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A NEW KIND OF COMPANION MATRIX*

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Abstract. A new kind of companion matrix is introduced, for polynomials of the form $c(\lambda) = \lambda a(\lambda)b(\lambda) + c_0$, where upper Hessenberg companions are known for the polynomials $a(\lambda)$ and $b(\lambda)$. This construction can generate companion matrices with smaller entries than the Fiedler or Frobenius forms. This generalizes Piers Lawrence's Mandelbrot companion matrix. The construction was motivated by use of Narayana-Mandelbrot polynomials, which are also new to this paper.

Key words. Companion matrix, Eigenvalue, Narayana's cows sequence, Mandelbrot polynomials, Mandelbrot matrices.

AMS subject classifications. 15A18, 15A23, 65F15, 65F50.

1. Introduction.

Recently, we generalized the Mandelbrot polynomials

$$p_{n+1} = zp_n^2 + 1, \quad p_0 = 0$$

to the Fibonacci-Mandelbrot polynomials

$$q_{n+1} = zq_nq_{n-1} + 1, \quad q_0 = 0, q_1 = 1$$

and generalized Piers Lawrence's supersparse[1] companion matrix for p_n [8] to an analogous one for q_n. See [4], [5] and [7] for details, though we summarize these constructions below.

If $p_n = \det(z\mathbf{I} - \mathbf{M}_n)$ for the Mandelbrot polynomials, then the subdiagonals of $\mathbf{M}_n$ are all -1 and the matrices are the same size, which gives

$$\mathbf{M}_{n+1} = \begin{bmatrix} \mathbf{M}_n & & -\mathbf{c}_n\mathbf{r}_n \\ -\mathbf{r}_n & 0 & \\ & -\mathbf{c}_n & \mathbf{M}_n \end{bmatrix}, \tag{1.1}$$

where $\mathbf{r}_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$ and $\mathbf{c}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ are both of length d_n. This is Piers Lawrence's original construction [8]. These are remarkable matrices: They contain only -1 or 0, and therefore are Bohemian matrices[2]; yet the characteristic polynomial contains coefficients that grow exponentially in the degree d_n (doubly exponentially in n).

For the Fibonacci-Mandelbrot polynomials, the degree of $q_n = F_n - 1$ and the construction contains matrices of different size. We begin with

$$\mathbf{M}_3 = \begin{bmatrix} -1 \end{bmatrix}$$

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[1] A matrix is supersparse if it is sparse and its nonzero elements are drawn from a small set, e.g. $\{-1, 1\}$.

[2] The name "Bohemian" is an acronym for Bounded height matrix of integers. See example OEIS A272658.

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http://repository.uwyo.edu/ela

and

$$\mathbf{M}_4 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

to construct our recursive companion matrix:

$$\mathbf{M}_{n+1} = \begin{bmatrix} \mathbf{M}_n & & (-1)^{d_{n+1}}\mathbf{c}_n\mathbf{r}_{n-1} \\ -\mathbf{r}_n & 0 & \\ & -\mathbf{c}_{n-1} & \mathbf{M}_{n-1} \end{bmatrix},$$

where $\mathbf{r}_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$ and $\mathbf{c}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ are, as before, the row and column vectors of length d_n. This gives a matrix of slightly greater height than (1.1) because the entries may be $\{-1, 0, 1\}$.

The surprising analogy between these two families of supersparse companions led us to conjecture and prove the following.

2. Main result.

THEOREM 2.1. *Suppose* $a(z) = \det(z\mathbf{I} - \mathbf{A})$, $b(z) = \det(z\mathbf{I} - \mathbf{B})$, *and both* $\mathbf{A}$ *and* $\mathbf{B}$ *are upper Hessenberg matrices with nonzero subdiagonal entries, and*

$$\alpha = \frac{1}{\left(\prod_{j=1}^{d_a-1} a_{j+1,j}\right)\left(\prod_{j=1}^{d_b-1} b_{j+1,j}\right)}$$

is the reciprocal of the product of the subdiagonal entries of $\mathbf{A}$ *and* $\mathbf{B}$, *and* $d_a = \deg_z a$ *and* $d_b = \deg_z b$, *so the dimension of* $\mathbf{A}$ *is* $d_a \times d_a$ *and the dimension of* $\mathbf{B}$ *is* $d_b \times d_b$. *Suppose both* d_a *and* d_b *are at least 1. Then if*

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & & -\alpha c_0 \mathbf{c_a}\mathbf{r_b} \\ -\mathbf{r_a} & 0 & \\ & -\mathbf{c_b} & \mathbf{B} \end{bmatrix},$$

where $\mathbf{r_a} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$ *of length* d_a *and* $\mathbf{c_b} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ *of length* d_b, *we have*

$$c(z) = \det(z\mathbf{I} - \mathbf{C}) = z \cdot a(z)b(z) + c_0.$$

REMARK 2.2. Proving this theorem automatically proves the validity of the constructions of the supersparse companion matrices for p_n, q_n, and r_n.

REMARK 2.3. Starting with a polynomial $c(z)$, we see that there are potentially many such $a(z)$ and $b(z)$. This freedom may be quite valuable or, it may be an obstacle.

Proof. Partition

$$z\mathbf{I} - \mathbf{C} = \left[\begin{array}{c|c} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \hline \mathbf{C}_{21} & \mathbf{C}_{22} \end{array}\right],$$

where $\mathbf{C}_{22} = z\mathbf{I} - \mathbf{B}$ is nonsingular if z is not an eigenvalue of $\mathbf{B}$, i.e., $b(z) \neq 0$. Later we will remove this

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restriction. Also,

$$\mathbf{C}_{21} = \begin{bmatrix} & & 1 \\ & & \\ & & \end{bmatrix}$$

is $d_b \times (d_a + 1)$ and has only one nonzero element, which is a 1 in the upper right corner. Next,

$$\mathbf{C}_{12} = \begin{bmatrix} & & \alpha c_0 \\ & & \\ & & \end{bmatrix}$$

is $(1 + d_a) \times d_b$ and again has only one nonzero element, αc_0 in the upper right corner. (In fact, c_0 can be zero.) This leaves

$$\mathbf{C}_{11} = \left[\begin{array}{c:c} & 0 \\ z\mathbf{I} - \mathbf{A} & \vdots \\ & 0 \\ & 0 \\ \hdashline \quad\quad 1 & z \end{array}\right],$$

which is $d_a + 1$ by $d_a + 1$.

The Schur factoring is

$$\begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{C}_{12} \\ 0 & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21} & 0 \\ \mathbf{C}_{22}^{-1}\mathbf{C}_{21} & \mathbf{I} \end{bmatrix},$$

with the computation of the Schur complement $\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}$ going to do most of the work in the proof. The Schur determinantal formula [10, Chapter 12] is then

$$\det \mathbf{C} = \det(\mathbf{C}_{22}) \det\left(\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right).$$

We have the following propositions.

1. $z\mathbf{I} - \mathbf{A}$ and $z\mathbf{I} - \mathbf{B}$ are upper Hessenberg because $\mathbf{A}$ and $\mathbf{B}$ are.
2. The first d_a columns of $\mathbf{C}_{22}^{-1}\mathbf{C}_{21}$ are zero.
3. The final column of $\mathbf{C}_{22}^{-1}\mathbf{C}_{21}$ is the solution, say $\vec{v}$, of $(z\mathbf{I} - \mathbf{B})\,\vec{v} = \mathbf{e}_1$. Again, $z\mathbf{I} - \mathbf{B}$ is nonsingular.
4. By Cramer's rule, the final entry in $\vec{v}$, say v, is

$$v = \frac{\det\left(\mathbf{C}_{22} \underset{d_b}{\leftarrow} \mathbf{e}_1\right)}{\det(\mathbf{C}_{22})},$$

where the notation $\mathbf{M} \underset{k}{\leftarrow} \vec{v}$ means replace the kth column of $\mathbf{M}$ with the vector $\vec{v}$ [3].

Electronic Journal of Linear Algebra, ISSN 1081-3810
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5. Since $\mathbf{C}_{22} = z\mathbf{I} - \mathbf{B}$ is upper Hessenberg,

$$\mathbf{C}_{22} \underset{d_b}{\longleftarrow} e_1 = \begin{bmatrix} * & * & * & \cdots & * & 1 \\ -b_{21} & * & * & \cdots & * & 0 \\ & -b_{32} & * & & \vdots & \vdots \\ & & -b_{43} & \ddots & & \\ & & & \ddots & & \\ & & & & * & 0 \\ & & & & -b_{d_b,d_b-1} & 0 \end{bmatrix}.$$

Laplace expansion about the final column gives

$$\begin{aligned} \det\left(\mathbf{C}_{22} \underset{d_b}{\longleftarrow} \mathbf{e}_1\right) &= (-1)^{d_b-1}(-1)^{d_b-1}\prod_{j=1}^{d_b-1} b_{j+1,j} \\ &= \prod_{j=1}^{d_b-1} b_{j+1,j}. \end{aligned}$$

Therefore,

$$v = \frac{\prod_{j=1}^{d_b-1} b_{j+1,j}}{b(z)}$$

because $\det \mathbf{C}_{22} = \det(z\mathbf{I} - \mathbf{B}) = b(z)$ by hypothesis.

6. Now

$$\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21} = \begin{bmatrix} & \alpha c_0 \\ & \\ & \end{bmatrix} \begin{bmatrix} * \\ \vdots \\ * \\ v \end{bmatrix} = \begin{bmatrix} & \alpha c_0 v \\ & \\ & \end{bmatrix}$$

is $d_a + 1$ by $d_a + 1$ and has its only nonzero entry, $\alpha c_0 v$, in the upper right corner.

7. The Schur complement is therefore

$$\left[\begin{array}{cccc|c} & & & & -\alpha c_0 v \\ & & & & 0 \\ & z\mathbf{I} - \mathbf{A} & & & \vdots \\ & & & & 0 \\ \hline 0 & \cdots & 0 & 1 & z \end{array}\right],$$

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http://repository.uwyo.edu/ela

and we compute $\det\left(\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right)$ by Laplace expansion on the last column:

$$\begin{aligned}\det\left(\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right) &= -(-1)^{d_a}\alpha c_0 v \det\begin{bmatrix} -a_{21} & * & * & \cdot & * \\ & -a_{32} & * & & * \\ & & -a_{43} & & \vdots \\ & & & \ddots & \\ & & & & -a_{d_a,d_a-1}\end{bmatrix} \\ &\quad + z\det\left(z\mathbf{I} - \mathbf{A}\right) \\ &= -(-1)^{d_a}\alpha c_0 v \prod_{j=1}^{d_a-1}\left(-a_{j+1,j}\right) + z\cdot a(z) \\ &= \alpha v \prod_{j=1}^{d_a-1} a_{j+1,j}\cdot c_0 + z\cdot a(z) \\ &= \alpha\cdot\frac{\left(\prod_{j=1}^{d_b-1} b_{j+1,j}\right)}{b(z)}\cdot\left(\prod_{j=1}^{d_a-1} a_{j+1,j}\right)\cdot c_0 + z\cdot a(z) \\ &= \frac{c_0}{b(z)} + z\cdot a(z)\end{aligned}$$

by the definition of α.

Therefore, by the Schur determinantal formula,

$$\begin{aligned}\det\left(z\mathbf{I} - \mathbf{C}\right) &= \det\left(\mathbf{C}_{22}\right)\det\left(\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right) \\ &= b(z)\left(\frac{c_0}{b(z)} + z\cdot a(z)\right) \\ &= z\cdot a(z)b(z) + c_0.\end{aligned}$$

Since the left hand side is a polynomial as is the right hand side, the formula will be true even if $b(z) = 0$, by continuity. □

3. Applications and examples. Sequence A000930 of the Online Encyclopedia of Integer Sequences, Narayana's cows sequence, begins

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, \ldots$$

and is generated by $R_n = R_{n-1} + R_{n-3}$ [13]. The connection to cows is that an ideal cow produces a calf every year, starting in its fourth year. Narayana was a mathematician in 14th century India. Various facts are known for this sequence, which is similar to the Fibonacci sequence: For instance, the generating function is $1/(1 - x - x^3)$. Many references are given in the OEIS, but see also [12].

We define the Narayana-Mandelbrot polynomials by $r_0 = 1, r_1 = r_2 = 1$ and

$$r_{n+1} = z r_n r_{n-2} + 1$$

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for $n \geq 2$. We construct a recursive family of companion matrices $\mathbf{R}_n$, i.e., such that

$$r_n(z) = \det(z\mathbf{I} - \mathbf{R}_n).$$

Just as the Fibonacci-Mandelbrot polynomials, the construction contains matrices of different sizes. However, for this family, we start with

$$\mathbf{R}_3 = \begin{bmatrix} -1 \end{bmatrix},$$

$$\mathbf{R}_4 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},$$

and

$$\mathbf{R}_5 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}.$$

Our construction is then

$$\mathbf{R}_{n+1} = \begin{bmatrix} \mathbf{R}_n & & (-1)^{d_{n+1}}\mathbf{c}_n\mathbf{r}_{n-2} \\ -\mathbf{r}_n & 0 & \\ & -\mathbf{c}_{n-2} & \mathbf{R}_{n-2} \end{bmatrix},$$

where $\mathbf{r}_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$ and $\mathbf{c}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ are, as before, the row and column vectors of length $d_n = \deg r_n = R_{n+1} - 1$.

This construction also allows new matrix families. For instance, suppose $s_0 = 0$, $s_{n+1} = z^3 s_n^4 + 1$. Then if $\mathbf{S}_n$ is an upper Hessenberg companion for s_n (with all -1 on the subdiagonal) the matrix

$$\mathbf{S_{n+1}} = \begin{bmatrix} \mathbf{S}_n & & & & & & & -\mathbf{c}_n\mathbf{r}_n \\ -\mathbf{r}_n & 0 & & & & & & \\ & -\mathbf{c}_n & \mathbf{S}_n & & & & & \\ & & -\mathbf{r}_n & 0 & & & & \\ & & & -\mathbf{c}_n & \mathbf{S}_n & & & \\ & & & & -\mathbf{r}_n & 0 & & \\ & & & & & -\mathbf{c}_n & \mathbf{S}_n & \end{bmatrix}$$

is an upper Hessenberg companion for s_{n+1}.

4. Concluding remarks. This is a genuinely new kind of companion matrix. We demonstrate this on Newton's example polynomial $x^3 - 2x - 5$. We see that $x^3 - 2x - 5 = x(x^2 - 2) - 5 = x(x - \sqrt{2})(x + \sqrt{2}) - 5$, and companion matrices for $x - \sqrt{2}$ and $x + \sqrt{2}$ are just $[+\sqrt{2}]$ and $[-\sqrt{2}]$ respectively. Thus, a companion matrix for Newton's polynomial is

$$\begin{bmatrix} \sqrt{2} & & 5 \\ -1 & & \\ & -1 & -\sqrt{2} \end{bmatrix}.$$

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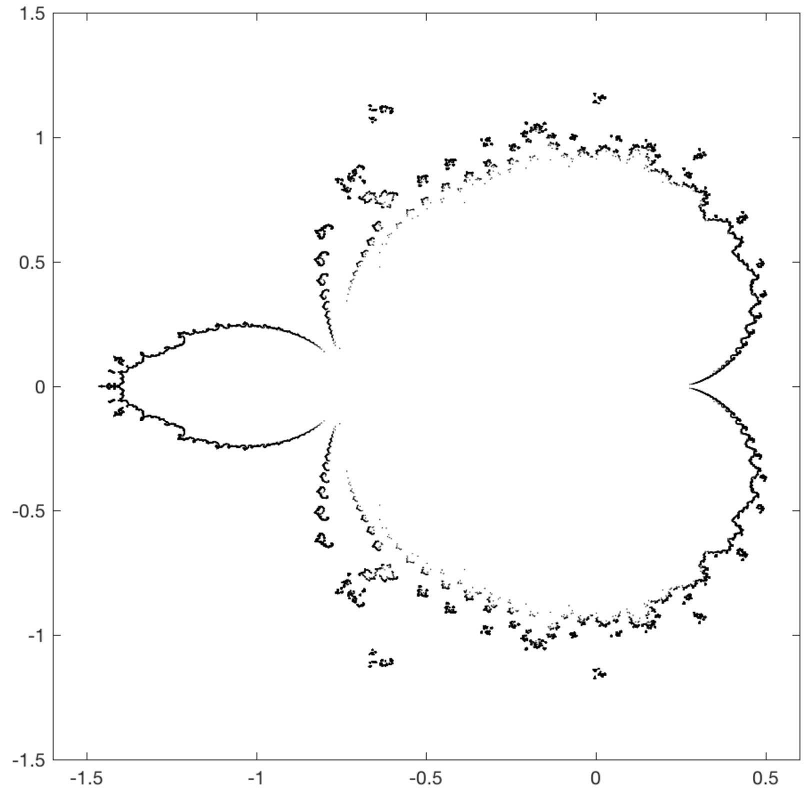

FIGURE 1. *Roots of Narayana-Mandelbrot polynomial,* $r_{36}(z)$*. The degree of* $r_{36}(z)$ *is 578,948.*

This matrix contains $\sqrt{5}$, unlike any previously recorded companion matrix. For unimodular polynomials, such companion matrices may be of lower height than the Frobenius or Fiedler [9] companions, and may offer better numerical condition.

We have now established that if $c(z) = z \cdot a(z)b(z) + c_0$ and $\mathbf{A}$ and $\mathbf{B}$ are upper Hessenberg companion matrices for the polynomials $a(z)$ and $b(z)$ respectively, then

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & & -\alpha c_0 \mathbf{c_a} \mathbf{r_b} \\ -\mathbf{r_a} & 0 & \\ & -\mathbf{c_b} & \mathbf{B} \end{bmatrix}$$

is a companion matrix for $c(z)$. One wonders immediately about a corresponding linearization, $\mathbf{L_C}$, strong or otherwise, for the matrix polynomial

$$\mathbf{C}(z) = z\mathbf{A}(z)\mathbf{B}(z) + \mathbf{C_0} \ ,$$

if $\mathbf{L_A}$ is a linearization for $\mathbf{A}$, $\mathbf{L_B}$ for $\mathbf{B}$. Some very preliminary experiments, where $\mathbf{L_A}$ and $\mathbf{L_B}$ were block

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http://repository.uwyo.edu/ela

upper Hessenberg with all blocks $\mathbf{I}$, so $\alpha = 1$, find that indeed

$$\mathbf{L_C} = \begin{bmatrix} \mathbf{L_A} & & & -\mathbf{C}_0 \\ & -\mathbf{I} & 0 & \\ & & -\mathbf{I} & \\ & & & \mathbf{L_B} \end{bmatrix}$$

is a (strong) linearization for $c(z)$, in the examples we tried.

In a paper to be submitted soon, we have now proved that this construction can be extended to matrix polynomials; see [6].

A referee pointed out that Robol et al. [11] use a similar construction to linearize polynomials of the form $p(z) = a(z)b(z) + zc(z)d(z)$ to find the roots of rational functions, which can also be applied to matrix polynomials.

We leave these extensions to future work.

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