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DETERMINANTAL REPRESENTATIONS OF ELLIPTIC CURVES VIA WEIERSTRASS ELLIPTIC FUNCTIONS*

MAO-TING CHIEN† AND HIROSHI NAKAZATO‡

Abstract. Helton and Vinnikov proved that every hyperbolic ternary form admits a symmetric derminantal representation via Riemann theta functions. In the case the algebraic curve of the hyperbolic ternary form is elliptic, the determinantal representation of the ternary form is formulated by using Weierstrass $\wp$-functions in place of Riemann theta functions. An example of this approach is given.

Key words. Determinantal representation, Elliptic curves, Weierstrass $\wp$-functions, Riemann theta functions.

AMS subject classifications. 15A60, 14H52.

1. Introduction. For an $n \times n$ complex matrix A, the associated derminantal ternary form $F_A(x, y, z)$ is defined by

$$F_A(x, y, z) = \det(x\Re(A) + y\Im(A) + zI_n),$$

where $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/(2i)$ are components of the Cartesian decomposition $A = \Re(A) + i\Im(A)$. The form $F_A(x, y, z)$ is hyperbolic, i.e., $F_A(x_0, y_0, z) = 0$ has n real roots counting the multiplicities for any $(x_0, y_0) \in \mathbb{R}^2$, and $F_A(0, 0, 1) = 1 \neq 0$. In 1981, Fiedler [3] conjectured the converse is true, namely, every hyperbolic ternary form admits a determinantal ternary form. He posed this conjecture to determine the numerical range $W(A)$ of A defined by

$$W(A) = \{\xi^* A\xi : \xi \in \mathbb{C}^n, \xi^*\xi = 1\}.$$

Kippenhahn [7] characterized $W(A)$ as the convex hull of the real affine part $\{x+iy : (x, y) \in \mathbb{R}^2, G(x, y, 1) = 0\}$ of the dual curve $G(x, y, z) = 0$ of the algebraic curve $F_A(x, y, z) = 0$.

Recently, Plaumann and Vinzant [12] proved that a hyperbolic ternary form $F(x, y, z)$ of degree n satisfying $F(0, 0, 1) = 1$ admits a derminantal representation

$$F(x, y, z) = \det(xH + yK + zI_n)$$

via Hermitian matrices H, K. Their proof is rather elementary. Historically, Lax [8] conjectured an assertion more strict than that of Fiedler in the sense that a hyperbolic ternary form $F(x, y, z)$ admits a derminantal representation

$$F(x, y, z) = \det(xS_1 + yS_2 + zI_n)$$

for some real symmetric matrices S_1 and S_2. Vinnikov [14, Theorem 6.1] proved that the Lax conjecture is true in the case where $F(x, y, z) = 0$ has no singular points. Helton and Vinnikov [5] used Riemann theta

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functions with characteristics to construct the symmetric matrices S_1 and S_2 . In fact, Fidler [4] already found a similar formula in the case the curve $F(x,y,z)=0$ is an irreducible rational curve, that is, the curve has a parametrization

$$x=u(s),\quad y=v(s),\quad z=w(s)$$

by three polynomials in one variable s. In [2], the authors of this note reformulated the Helton-Vinnikov formula [5] when $F(x,y,z)=0$ is an irreducible curve with genus $g=0$, a rational curve, and $g=1$, an elliptic curve. The constructions of the real symmetric matrices in both papers [2] and [5] base heavily on computing the Riemann theta functions of a Riemann surface. A Riemann surface is a one-complex-dimensional connected complex analytic manifold [10, 13]. The Riemann theta functions may encounter complexity in computation. In this note, we propose the Weierstrass $\wp$-functions to formulate the real symmetric matrices for hyperbolic ternary forms of elliptic curves. An illustrative example is given for constructing the symmetric matrices of an elliptic hyperbolic ternary form via Weierstrass $\wp$-functions. The computation method involves the abelian group structure of the elliptic curve.

2. Weierstrass $\wp$-function method. Let $F(x,y,z)$ be an irreducible ternary form of degree $n\geq 2$. A point $P_0=(x_0,y_0,z_0)\in\mathbb{C}^3\backslash\{(0,0,0)\}$ of the complex projective curve $F(x,y,z)=0$ is called a singular point if

$$\frac{\partial F}{\partial x}(x_0,y_0,z_0)=\frac{\partial F}{\partial y}(x_0,y_0,z_0)=\frac{\partial F}{\partial z}(x_0,y_0,z_0)=0.$$

For a singular point $P_0=(x_0,y_0,z_0), z_0\neq 0$, consider two functions

$$f(X,Y)=F(x_0+X,y_0+Y,z_0)\quad\text{and}\quad g(X,Y)=F_y(x_0+X,y_0+Y,z_0).$$

The Taylor series of the two functions define an ideal (f,g) of the ring $\mathbb{C}[[X,Y]]$ of formal power series in X,Y. We define

$$\delta(P_0)=\frac{1}{2}\left(\dim\left(\frac{\mathbb{C}[[X,Y]]}{(f,g)}\right)-m+s\right),$$

where m is the multiplicity of P_0 and s is the number of irreducible analytic branches of the curve $F(x,y,z)=0$ near (x_0,y_0,z_0). The number $\delta(P_0)$ can be evaluated by Maple package for ternary forms with integral coefficients when the number of terms is not large. The genus of an irreducible curve $F(x,y,z)=0$ of degree n is defined by

$$g(F)=\frac{1}{2}(n-1)(n-2)-\sum_{j=1}^{k}\delta(P_j),$$

where $P_1,\ldots,P_k$ are singular points of the curve $F(x,y,z)=0$. An irreducible curve is called an *elliptic* curve if its genus is $g=1$. The algebraic curve of the determinantal ternary form associated to a cyclic weighted shift matrix with suitable weights is a typical elliptic curve. It is proved in [1] that hyperbolic ternary forms satisfying weakly symmetry admit determinantal representations via cyclic weighted shift matrices for lower degrees under the assumption that the curve $F(x,y,z)=0$ is an irreducible elliptic curve. Lentzos and Pasley [9] solved the problem for general degrees without any restriction on the genus of the associated curve.

Let $F(x,y,z)$ be a real irreducible hyperbolic ternary form of degree n with respect to $(0,0,1)$ and the curve $F(x,y,z)=0$ is elliptic. We assume that $F(0,0,1)=1$ and the equation $F(0,-1,z)=0$ has n distinct non-zero roots $\beta_1,\beta_2,\ldots,\beta_n$. Then there is a real bi-rational transformation Φ which transforms the elliptic curve $F(x,y,z)=0$ to a non-singular cubic curve of the Weierstrass standard form

$$Y^2Z=4X^3-g_2XZ^2-g_3Z^3=4(X-e_1Z)(X-e_2Z)(X-e_3Z)$$

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for some real constants g_2, g_3 satisfying $g_2^3 - 27g_3^2 > 0$ and some real roots $e_1 > e_2 > e_3$ with $e_1 + e_2 + e_3 = 0$. We consider the elliptic integrals

$$\omega_1 = \int_{e_1}^{+\infty} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} = \int_{e_3}^{e_2} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} > 0,$$

and

$$\frac{\omega_2}{i} = \int_{-\infty}^{e_3} \frac{dx}{\sqrt{-4x^3 + g_2 x - g_3}} = \int_{e_2}^{e_1} \frac{dx}{\sqrt{-4x^3 + g_2 x + g_3}} > 0.$$

Further, we define a lattice $\Gamma \subset \mathbb{C}$ by

$$\Gamma = \{2n_1\omega_1 + 2n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}.$$

The Weierstrass elliptic function (or Weierstrass $\wp$-function) $\wp(s) = \wp(s; g_2, g_3)$ with fundamental half-periods $\omega_1 > 0$ and $\omega_2 \in i\mathbb{R}$, $\Im(\omega_2) > 0$ in the sense $\wp(s + 2\omega_1) = \wp(s)$ and $\wp(s + 2\omega_2) = \wp(s)$, is defined as a solution of the ordinary differential equation

$$\left(\frac{d\wp(s)}{ds}\right)^2 = 4\wp(s)^3 - g_2\wp(s) - g_3,$$

for which the Laurent series expansion of $\wp(s)$ near the pole $s = 0$ is

$$\wp(s; g_2, g_3) = \frac{1}{s^2} + \frac{1}{20} g_2 s^2 + \frac{1}{28} g_3 s^4 + \cdots.$$

The second derivative $\wp''(s)$ of the $\wp$-function satisfies

$$\frac{d^2\wp(s)}{ds^2} = 6\wp(s)^2 - \frac{1}{2} g_2.$$

(For reference on elliptic curves, see, for instance, [6, Chapter 1].)

Denote the upper half-plane $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. The Riemann theta function is the holomorphic function on $\mathbb{C} \times \mathcal{H}$ defined by the exponential series

$$\theta(u, \tau) = \sum_{m \in \mathbb{Z}} \exp(\pi i(m^2\tau + 2mu)),$$

which is quasi-periodic with respect to the lattice $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$:

$$\theta(u + m + \tau n, \tau) = \exp(\pi i(-2nu - n^2\tau))\, \theta(u, \tau)$$

for all integers m, n. We consider four Riemann theta functions $\theta[\epsilon](u)$ with characteristics ϵ defined by

$$\theta[\epsilon](u, \tau) = \exp(\pi i(a^2\tau + 2au + 2ab))\theta(u + \tau a + b, \tau)$$

for $\epsilon = a + \tau b$ with

$$(a, b) = (0, 0),\ (1/2, 0),\ (0, 1/2),\ (1/2, 1/2).$$

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Using the parameter $q = \exp(i\pi\tau)$, the four Riemann theta functions θ_δ, $\delta = 1, 2, 3, 4$, corresponding to $\epsilon = a + \tau b$, $(a, b) = (1/2, 1/2), (1/2, 0), (0, 0), (0, 1/2)$, are respectively defined by

$$\theta_1(u) = -\theta[1/2 + \tau(1/2)](u, \tau) = 2q^{1/4} \sum_{m=0}^{\infty} q^{(m+1)m} \sin((2m + 1)\pi u),$$

$$\theta_2(u) = \theta[1/2 + \tau 0](u, \tau) = 2q^{1/4} \sum_{m=0}^{\infty} q^{(m+1)m} \cos((2m + 1)\pi u),$$

$$\theta_3(u) = \theta[0 + \tau 0](u, \tau) = \theta(u, \tau) = 1 + 2 \sum_{m=1}^{\infty} q^{m^2} \cos(2m\pi u),$$

$$\theta_4(u) = \theta[0 + \tau(1/2)](u, \tau) = 1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cos(2m\pi u).$$

The meromorphic functions defined by

$$\sqrt{\wp(s; g_2, g_3) - e_k}, \quad k = 1, 2, 3$$

depend on their expressions by θ-functions. We observe that the three functions

$$\wp(s) - e_3 > \wp(s) - e_2 > \wp(s) - e_1$$

take non-negative real values on the real line $\Im(s) = 0$ satisfying $\wp(2n\omega_1) = \infty$ for every integer n. On the line $\Im(s) = \Im(\omega_2)$, the function $\wp(s)$ takes real values satisfying $e_3 \leq \wp(s) \leq e_2$. In [6, Chapter 2], the functions $\sqrt{\wp(s) - e_k}$, $k = 1, 2, 3$, are treated by using θ-functions. We remark that

$$2\omega_1 > 0, \quad \theta_1'(0) > 0, \quad \theta_2(0) > 0, \quad \theta_3(0) > 0, \quad \theta_4(0) > 0.$$

The behavior of the following three elliptic functions

$$\frac{\theta_2(u)}{\theta_1(u)}, \quad \frac{\theta_3(u)}{\theta_1(u)}, \quad \frac{\theta_4(u)}{\theta_1(u)}$$

on the line $\Im(u) = 0$ and the line $\Im(u) = \Im(\tau)/2$ is useful for the symmetric matrices constructed by Helton and Vinnikov [5], where $\tau = \omega_2/\omega_1$. The functions $\theta_\delta(u)/\theta_1(u)$ have fundamental periods $1, 2\tau$ ($\delta = 2$), $2, 2\tau$ ($\delta = 3$) and $2, \tau$ ($\delta = 4$) [2]. Note that two periods p_1 and p_2 of an elliptic function are called fundamental periods if $\Im(p_2/p_1) \neq 0$ and any other period p can be written as $p = mp_1 + np_2$ for some integers m and n.

The Helton-Vinnikov theorem [5, Theorem 2.2] of the symmetric determinantal representation of a hyperbolic ternary form involves Riemann theta functions. We reformulate Helton-Vinnikov theorem for the symmetric determinantal representation of a hyperbolic ternary form via Weierstrass $\wp$-functions. Firstly, we give the determinantal representation in terms of Riemann theta functions.

THEOREM 2.1. (cf. [2, Theorem 2.4]) *Let $F(x, y, z)$ be an irreducible real hyperbolic ternary form of degree n with respect to $(0, 0, 1)$ and satisfying $F(0, 0, 1) = 1$ and $F(0, -1, z) = 0$ has n distinct non-zero real roots $\beta_1, \beta_2, \ldots, \beta_n$. Assume that the curve $F(x, y, z) = 0$ is elliptic which is birationally transformed to the cubic curve $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$, and the affine curve $F(x, y, 1) = 0$ is parametrized by $x = R_1(\wp(s), \wp'(s)), y = R_2(\wp(s), \wp'(s))$ for some real functions $R_1(u, v)$ and $R_2(u, v)$. Further, assume $2\omega_1 \in \mathbb{R}^+$ and $2\omega_2 \in i\mathbb{R}^+$ are the fundamental periods of the elliptic function $\wp(s; g_2, g_3)$. Let Q_j be the point*

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of the torus $\mathbb{C}/(2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2)$ *corresponding to the point* $(0, -1, \beta_j)$ *of the curve* $F(x, y, z) = 0$. *Then the ternary form* $F(x, y, z)$ *has two determinantal representations*

$$F(x, y, z) = \det(x\, C_\delta + y\, \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n) + z\, I_n), \quad \delta = 2, 3 \tag{2.1}$$

for some real symmetric matrices C_2 *and* C_3 *which have common diagonal entries*

$$c_{jj} = \beta_j \frac{F_x(0, -1, \beta_j)}{F_y(0, -1, \beta_j)}, \quad j = 1, 2, \ldots, n \tag{2.2}$$

and off-diagonal real entries given by

$$c_{jk} = \frac{(\beta_k - \beta_j)\theta_1'(0)}{2\omega_1\, \theta_\delta(0)} \cdot \frac{\theta_\delta((Q_k - Q_j)/(2\omega_1))}{\theta_1((Q_k - Q_j)/(2\omega_1))} \cdot \frac{1}{\sqrt{d(R_1/R_2)(Q_j)}\sqrt{d(R_1/R_2)(Q_k)}}. \tag{2.3}$$

Proof. The diagonals (2.2) of C_δ can be easily derived from the representation (2.1). The off diagonals of C_δ based on the Riemann theta functions given in [5, Theorem 2.2](also [11, Theorem 6]) for hyperbolic algebraic curves with arbitrary genus g are expressed by

$$c_{jk} = \frac{\beta_k - \beta_j}{\theta[\delta](0)} \cdot \frac{\theta[\delta](Q_k - Q_j, S)}{E(Q_k, Q_j)} \cdot \frac{1}{\sqrt{d(x/y)(Q_j)}\sqrt{d(x/y)(Q_k)}}, \tag{2.4}$$

where $\theta[\delta](\cdot,\cdot)$ is a Riemann theta function with an even characteristic δ, $E(\cdot,\cdot)$ is the prime form on the Jacobi-variety given as a constant multiple of a Riemann theta function $\theta[\epsilon](\cdot,\cdot)$ with an odd characteristic ϵ, the two Riemann theta functions are defined for $(z, S) \in \mathbb{C}^g \times \mathcal{H}_g$, S is Riemann's period matrix with respect to the canonical basis $\{\omega_1, \ldots, \omega_g\}$ of holomorphic differential 1-forms on the Riemann surface determined by the curve $F(x, y, z) = 0$ (cf. [10, 13]). The set $\mathcal{H}_g$ denotes the space of $g \times g$ symmetric matrices whose imaginary parts are positive definite.

For $g = 1$, the formula (2.4) is reformulated in [2, Theorem 2.4] and is given by (2.3). The points Q_k, $k = 1, 2, \ldots, n$, of the torus $\mathbb{C}/(2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2)$ satisfy $\Im(Q_k) \cong 0$ or $\Im(Q_k) \equiv \Im(\omega_2)$ modulo $2\Im(\omega_2)$. On the torus $\mathbb{C}/(2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2)$, the equivalence relation $z_1 \equiv z_2$ means $z_1 - z_2 = 2n\omega_1 + 2m\omega_2$ for some integers n, m. The derivative $d(R_1/R_2)(Q_k)$ is always real valued, and the value $\sqrt{d(R_1/R_2)(Q_j)}\sqrt{d(R_1/R_2)(Q_k)}$, $j \neq k$, is real if $\Im(Q_j - Q_k) \equiv 0$ and it is pure-imaginary if $\Im(Q_j - Q_k) \equiv \Im(\omega_2)$. This property of the function $\sqrt{d(R_1/R_2)(Q_k)}\sqrt{d(R_1/R_2)(Q_j)}$ and the properties of the functions

$$\frac{\theta_\delta([Q_k - Q_j]/(2\omega_1))}{\theta_1([Q_k - Q_j]/(2\omega_1))}, \quad \delta = 2, 3, 4$$

imply that the entries c_{jk} for $\delta = 2, 3$ are real, and pure imaginary for $\delta = 4$ and some $j \neq k$. ∎

Next, we approach the symmetric determinantal representation of a hyperbolic ternary form via Weierstrass $\wp$-functions.

THEOREM 2.2. *Let* $F(x, y, z)$ *be an irreducible real hyperbolic ternary form of degree* n *with respect to* $(0, 0, 1)$ *and satisfying* $F(0, 0, 1) = 1$ *and* $F(0, -1, z) = 0$ *has* n *distinct non-zero real roots* $\beta_1, \beta_2, \ldots, \beta_n$. *Assume that the curve* $F(x, y, z) = 0$ *is elliptic which is birationally transformed to the cubic curve* $Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3$, *and the affine curve* $F(x, y, 1) = 0$ *is parametrized by* $x = R_1(\wp(s), \wp'(s)), y = R_2(\wp(s), \wp'(s))$ *for some real functions* $R_1(u, v)$ *and* $R_2(u, v)$. *Further, assume* $2\omega_1 \in \mathbb{R}^+$ *and* $2\omega_2 \in i\mathbb{R}^+$ *are the fundamental periods of the elliptic function* $\wp(s; g_2, g_3)$. *Let* Q_j *be the point of the torus* $\mathbb{C}/(2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2)$

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corresponding to the point $(0,-1,\beta_j)$ of the curve $F(x,y,z)=0$. Then the ternary form $F(x,y,z)$ has two determinantal representations (2.1) with common diagonal entries (2.2) in Theorem 2.1, and off-diagonal entries given by

$$c_{jk}=(\beta_k-\beta_j)\frac{\epsilon\sqrt{\wp(Q_k-Q_j)-e_{\delta-1}}}{\sqrt{d(R_1/R_2)(Q_j)d(R_1/R_2)(Q_k)}},\quad \delta=2,3, \tag{2.5}$$

where $\epsilon=\pm1$ are signatures depending on j,k.

Proof. To achieve the formula (2.5), we apply the formula (2.3) and make the use of the identity in [6, Chapter 2] that

$$\left(\frac{\theta_1'(0)}{2\omega_1\theta_{k+1}(0)}\frac{\theta_{k+1}(u)}{\theta_1(u)}\right)^2=\wp(2\omega_1 u)-e_k,\quad k=1,2,3. \tag{2.6}$$

The function $\theta_4(u)/\theta_1(u)$ takes real values both on the line $\Im(u)=0$ and the line $\Im(u)=\Im(\tau/2)$, and satisfies $\theta_4(u+1)/\theta_1(u+1)=-\theta_4(u)/\theta_1(u)$. The functions $\theta_\delta/\theta_1(u),\delta=2,3$, take real values on the line $\Im(u)=0$ and take pure imaginary values on the line $\Im(u)=\Im(\tau/2)$. On the line $\Im(u)=0$, the three functions $\theta_k(u)/\theta_1(u)$, $k=2,3,4$, satisfy the common property

$$\lim_{u\to+0}\frac{\theta_2(u)}{\theta_1(u)}=\lim_{u\to+0}\frac{\theta_3(u)}{\theta_1(u)}=\lim_{u\to+0}\frac{\theta_4(u)}{\theta_1(u)}=+\infty.$$

The derivative of the function $\theta_2(u)/\theta_1(u)$ at $u=1/2$ is negative. The function $\wp(s)-e_1$ behaves as

$$\wp(\omega_1+s)-e_1=(e_1-e_2)(e_1-e_3)s^2+b_4s^4+\cdots$$

near the half-period ω_1. The functions $\theta_3(u)/\theta_1(u)$ and $\theta_4(u)/\theta_1(u)$ take strictly positive values on the line $\Im(u)=0$. Thus, the formula (2.6) implies that

$$\frac{\theta_1'(0)}{2\omega_1\theta_{k+1}(0)}\frac{\theta_{k+1}(u)}{\theta_1(u)}=\sqrt{\wp(2\omega_1u)-e_k},$$

$k=2,3$, on the line $\Im(u)=0$ where $\sqrt{v}$ is a positive square root of a positive real number v. We also obtain that

$$\frac{\theta_1'(0)}{2\omega_1\theta_2(0)}\frac{\theta_2(u)}{\theta_1(u)}=\sqrt{\wp(2\omega_1u)-e_1}$$

on the interval $0<u\le 1/2$, and

$$\frac{\theta_1'(0)}{2\omega_1\theta_2(0)}\frac{\theta_2(u)}{\theta_1(u)}=-\sqrt{\wp(2\omega_1u)-e_1}$$

on the interval $-1/2\le u<0$.

On the line $\Im(u)=\Im(\tau/2)$, the function $\theta_3(u)/\theta_1(u)$ has the properties: $\theta_3(u+1)/\theta_1(u+1)=-\theta_3(u)/\theta_1(u)$, $\frac{\theta_3}{\theta_1}(\frac{1}{2}+\frac{\tau}{2})=0$, and the derivative of the function $\Im(\theta_3(u)/\theta_1(u))$ at $u=1/2+\tau/2$ is positive. The behavior of the non-negative function $e_2-\wp(s)$ on the line $\Im(s)=\Im(\omega_2)$ near the point $s=\omega_1+\omega_2$ is expanded as

$$\wp(\omega_1+\omega_2+s)-e_2=-(e_1-e_2)(e_2-e_3)s^2+c_4s^4+\cdots.$$

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Hence, $e_2-\wp(\omega_1+\omega_2+s)$ is approximated to $(e_1-e_2)(e_2-e_3)s^2$ which is positive for real number s. Thus, on the interval $[-1/2+\tau/2, 3/2+\tau/2]$ and $k=2$, we have

$$\frac{\theta_1'(0)}{2\omega_1\theta_3(0)}\Im(\frac{\theta_3(u)}{\theta_1(u)})=\sqrt{e_2-\wp(2\omega_1 u)}$$

for $\Im(u)=\Im(\tau)/2$, $1/2\le\Re(u)\le 3/2$, and

$$\frac{\theta_1'(0)}{2\omega_1\theta_3(0)}\Im(\frac{\theta_3(u)}{\theta_1(u)})=-\sqrt{e_2-\wp(2\omega_1 u)}$$

for $\Im(u)=\Im(\tau)/2$, $-1/2\le\Re(u)\le 1/2$. The elliptic function $\theta_2(u)/\theta_1(u)$ has the properties: $\frac{\theta_2}{\theta_1}(u+\frac{\tau}{2})=-i\,\frac{\theta_3}{\theta_4}(u)$, $\Im(\frac{\theta_2}{\theta_1})(u)<0$ on the line $\Im(u)=\Im(\tau)/2$, and $\Im(\frac{\theta_2}{\theta_1})(u)>0$ on the line $\Im(u)=-\Im(\tau)/2$. These imply that

$$\frac{\theta_1'(0)}{2\omega_1\theta_2(0)}\Im(\frac{\theta_2(u)}{\theta_1(u)})=-\sqrt{e_1-\wp(2\omega_1 u)}$$

for $\Im(u)=\Im(\tau)/2$, and

$$\frac{\theta_1'(0)}{2\omega_1\theta_2(0)}\Im(\frac{\theta_2(u)}{\theta_1(u)})=\sqrt{e_1-\wp(2\omega_1 u)}$$

for $\Im(u)=-\Im(\tau)/2$.

The derivative of the real-valued function $\theta_4(u)/\theta_1(u)$ at $u=\tau/2$ is positive and negative at $u=1+\tau/2$. The function $\wp(\omega_2+s)-e_3$ behaves as

$$\wp(\omega_2+s)-e_3=(e_1-e_3)(e_2-e_3)s^2+d_4s^4+\cdots$$

near ω_2. Similarly, we obtain that

$$\frac{\theta_1'(0)}{2\omega_1\theta_4(0)}\frac{\theta_4(u)}{\theta_1(u)}=\sqrt{\wp(2\omega_1 u)-e_3}$$

for $\Im(u)=\Im(\tau)/2$, $0\le\Re(u)\le 1$, and

$$\frac{\theta_1'(0)}{2\omega_1\theta_4(0)}\frac{\theta_4(u)}{\theta_1(u)}=-\sqrt{\wp(2\omega_1 u)-e_3}$$

for $\Im(u)=\Im(\tau)/2$, $-1\le\Re(u)\le 0$. Thus, the function $\sqrt{\wp(Q_k-Q_j)-e_3}$ takes real value for any $k\ne j$. The functions $\sqrt{\wp(Q_k-Q_j)-e_\delta}$, $\delta=1,2$, take real values if $\Im(Q_k-Q_j)\equiv 0$ modulo $2\Im(\omega_2)$, and pure imaginary values if $\Im(Q_k-Q_j)\equiv\Im(\omega_2)$. This completes the proof. ∎

3. An example. We give an example to verify Theorem 2.2 for computing the symmetric matrices of an elliptic hyperbolic ternary form. Consider a 4×4 complex symmetric matrix $T(a,b)$ given by

$$T(a,b)=\begin{pmatrix} i(a+b) & 0 & a & b\\ 0 & -i(a+b) & b & a\\ a & b & i(a-b) & 0\\ b & a & 0 & -i(a-b)\end{pmatrix},$$

where a,b are non-zero real numbers and $b\ne\pm a$. Then $T(a,b)^4=-4(a^2-b^2)^2I_4$, and the associated ternary form $F(x,y,z)$ of $T(a,b)$ is given by

$$F(x,y,z)=z^4-2(a^2+b^2)(x^2+y^2)z^2+(a^2-b^2)^2(x^2-y^2)^2.$$

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The singular points of the curve $F(x,y,z)=0$ are $(1,1,0)$ and $(1,-1,0)$. Then the genus of the curve $F(x,y,z)=0$ is 1, the curve $F(x,y,z)=0$ is elliptic. For computation simplicity, we assume $b=1$. Consider a quadratic transformation

$$\begin{aligned} x &= \frac{a+1}{4}\Big((a+1)^2(X+Y)Y+Z^2\Big),\\ y &= \frac{(a+1)^2}{4}(X+2Y)Z,\\ z &= \frac{(a+1)^2}{4}\Big((a+1)(X+Y)-Z\Big)\Big((a+1)(X+Y)-Z\Big) \end{aligned}$$

(cf. [15]). The inverse quadratic transformation is given by

$$X=-x^2+y^2+\frac{z^2}{(a+1)^2},\quad Y=x^2-y^2+\frac{x\,z}{a+1},\quad Z=y\,z.$$

To construct a determinantal representation C_2 for the curve $F(x,y,z)=0$ in the case $a>1$, we simplify the coefficients of the polynomial $F(x,y,z)$ by multiplying the factor $2^8=256$, and substitute the above transformation into $256F(x,y,z)=0$. There obtains that

$$P(X,Y,Z)\Big((a+1)^4X^3+2(a+1)^4X^2Y+4a(a+1)^2XY^2-4(a^2+a+1)X^2Z-8(a^2+1)YZ^2\Big)=0,$$

where

$$P(X,Y,Z)=(a+1)^8(X+2Y)(X+aX+Y+aY-Z)^2(X+aX+Y+aY+Z)^2.$$

Disregard the exceptional lines, the elliptic curve $F(x,y,z)=0$ is transformed into the cubic curve

$$(a+1)^4X^3+2(a+1)^4X^2Y+4a(a+1)^2XY^2-4(a^2+a+1)X^2Z-8(a^2+1)YZ^2=0. \tag{7}$$

Changing the following variables

$$\begin{aligned} X &= \frac{(a^2+1)}{6(a^2+a+1)^3}\Big(-12(a^2+a+1)^4X_2+(a^4+1)(a+1)^2Z_2\Big),\\ Y &= \frac{1}{12(a^2+a+1)^3}\Big(12(a^2+a+1)^3X_2-(a+1)^2(a^3+a^2+1)(a^3+a+1)Z_2\Big),\\ Z &= Y_1, \end{aligned}$$

the cubic curve in Weierstrass standard form becomes

$$Y_1^2Z_2=4X_2^3-g_2X_2Z_2-g_3Z_2^3,$$

where

$$g_2=\frac{(a+1)^4(a^8-a^4+1)}{12(a^2+a+1)^4}\quad\text{and}\quad g_3=-\frac{(a+1)^6(a^4-2)(a^4+1)(2a^4-1)}{432(a^2+a+1)^6}.$$

The points $\tilde{\beta_1}=(0,-1,a+1)$, $\tilde{\beta_2}=(0,-1,-a-1)$, $\tilde{\beta_3}=(0,-1,a-1)$, $\tilde{\beta_4}=(0,-1,-a+1)$ on the curve $F(x,y,z)=0$ are transformed accordingly to $\tilde{Q}_1=(X,Y,Z)=(-2,1,a+1)$, $\tilde{Q}_2=(2,-1,a+1)$, $\tilde{Q}_3=\Big(-\frac{2(a^2+1)}{(a+1)^2},1,a-1\Big)$, $\tilde{Q}_4=\Big(\frac{2(a^2+1)}{(a+1)^2},-1,a-1\Big)$ on the curve (7) respectively. The point $(X,Y,Z)=(0,0,1)$ is a point of the reflection of the cubic group. We adopt the elliptic curve group structure with the neutral element at this point $(X,Y,Z)=(0,0,1)$. The cubic curve is symmetric with respect to the line $Z=0$. For

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any point $P = (X_0, Y_0, Z_0)$ of the cubic curve, the point $(X_0, Y_0, -Z_0)$ is a point of the cubic curve and this point is $-P$ with respect to the elliptic curve group structure. With this group structure, the point $\tilde{Q}_2$ is $-\tilde{Q}_1$, and the point $\tilde{Q}_4$ is $-\tilde{Q}_3$. We call that the point $2P = P + P = (X_{10}, Y_{10}, Z_{10})$ of the cubic curve for a point $P = (X_0, Y_0, Z_0)$ is characterized as the point $(X_{10}, Y_{10}, -Z_{10})$ on the cubic curve which is another intersection of the tangent line L of the cubic curve at the point P. By using this definition, we compute $2\tilde{Q}_j$ and $4\tilde{Q}_j$ for $j = 1, 2, 3, 4$, and obtain the relations:

$$\tilde{Q}_1 + \tilde{Q}_1 = \tilde{Q}_2 + \tilde{Q}_2 = \tilde{Q}_3 + \tilde{Q}_3 = \tilde{Q}_4 + \tilde{Q}_4 = [(0, 1, 0)].$$

The point $Q = (X, Y, Z) = (0, 1, 0)$ satisfies $Q + Q = [(0, 0, 1)]$, that is, $Q + Q = 0$ with respect to this elliptic group structure. The point Q lies on the pseudo-line part of the real part of the cubic curve. Thus, it corresponds to the point ω_1 of the torus $\mathbb{C}/\Gamma$. The point $\tilde{Q}_1+\tilde{Q}_3$ has the coordinates $(X, Y, Z) = (-2, a+1, 0)$. The point $\tilde{Q}_1 + \tilde{Q}_4 = (-2a, a + 1, 0)$.

In the coordinates (X_2, Y_1, Z_2), the point $\tilde{Q}_1 + \tilde{Q}_1 = \tilde{Q}_3 + \tilde{Q}_3$ is expressed as $(X_2, Y_1, Z_2) = (e_1, 0, 1)$, where

$$e_1 = \frac{(a + 1)^2(a^4 + 1)}{12(a^2 + a + 1)^1}.$$

Similarly, the points $\tilde{Q}_1 + \tilde{Q}_3$, $\tilde{Q}_1 + \tilde{Q}_4$ are respectively expressed as

$$(X_2, Y_1, Z_2) = (e_{1,3}, 0, 1) \quad \text{and} \quad (X_2, Y_1, Z_2) = (e_{1,4}, 0, 1),$$

where

$$e_{1,3} = \frac{(a + 1)^2(a^4 - 2)}{12(a^2 + a + 1)^2} \quad \text{and} \quad e_{1,4} = -\frac{(a + 1)^2(2a^4 - 1)}{12(a^2 + a + 1)^2}.$$

These quantities satisfy

$$e_1 > e_{1,3} > e_{1,4}$$

in the case $|a| > 1$, and

$$e_1 > e_{1,4} > e_{1,3}$$

in the case $-1 < a < 1, a \neq 0$. In the coordinates (X_2, Y_1, Z_2), the points $\tilde{Q}_1$, $\tilde{Q}_2$ are respectively given by

$$(X_2, Y_1, Z_2) = \left(\frac{(a + 1)^2(a^4 + 3a^2 + 1)}{12(a^2 + a + 1)^2}, \frac{a^2(a + 1)^3(a^2 + 1)}{4(a^2 + a + 1)^3}, 1\right)$$

and

$$(X_2, Y_1, Z_2) = \left(\frac{(a + 1)^2(a^4 + 3a^2 + 1)}{12(a^2 + a + 1)^2}, -\frac{a^2(a + 1)^3(a^2 + 1)}{4(a^2 + a + 1)^3}, 1\right).$$

Since

$$\frac{(a + 1)^2(a^4 + 3a^2 + 1)}{12(a^2 + a + 1)^2} > e_1,$$

these points belongs to the pseudo-line part of the real part of the cubic. In the case $a > -1$, the Y_1/Z_2-coordinates of $\tilde{Q}_1$ is greater than that of $\tilde{Q}_2$. In the case $a < -1$, the reverse inequality holds. By using the condition

$$\tilde{Q}_1 + \tilde{Q}_1 + \tilde{Q}_1 + \tilde{Q}_1 = [(0, 1, 0] + [(0, 1, 0)] = 0,$$

we conclude that the points Q_1, Q_2 on the torus $\mathbb{C}/\Gamma$ corresponding to $\tilde{Q}_1, \tilde{Q}_2$ are given by

$$Q_1 = \frac{3}{2}\omega_1 \quad \text{and} \quad Q_2 = \frac{1}{2}\omega_1 \cong -\frac{3}{2}\omega_1$$

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in the case $a > -1$, and

$$Q_1 = \frac{1}{2}\omega_1 \quad \text{and} \quad Q_2 = \frac{3}{2}\omega_1$$

in the case $a < -1$. We recall that $\wp(\omega_2) = e_3 < e_2 = \wp(\omega_1 + \omega_2)$. The points $Q_3 = -Q_4$ on the torus corresponding to points $\tilde{Q}_3$, $\tilde{Q}_4$ satisfy

$$Q_3 = \omega_2 + \frac{3}{2}\omega_1 \quad \text{and} \quad Q_4 = \omega_2 + \frac{1}{2}\omega_1$$

or

$$Q_3 = \omega_2 + \frac{1}{2}\omega_1 \quad \text{and} \quad Q_4 = \omega_2 + \frac{3}{2}\omega_1.$$

The function R_2/R_1 on the torus $\mathbb{C}/\Gamma$ is expressed as a rational function in $X_2/Z_2 = \wp(s)$ and $Y_1/Z_2 = \wp'(s)$:

$$\frac{R_2}{R_1} = \frac{1}{72a(a+1)(a^2+a+1)^3\wp'(s)}\left(-12(a^2+a+1)^2\wp(s) + (a+1)^2(a^2-a-1)(a^2+a-1)\right)$$
$$\times\left(-12(a^2+a+1)^2\wp(s) + (a+1)^2(a^4+3a^2+1)\right).$$

Moreover, its derivative $d(R_2/R_1)$ with respect to s is expressed as a rational function in $X_2/Z_2 = \wp(s)$:

$$d(R_2/R_1) = \frac{-1}{12a(a+1)(a^2+a+1)} \cdot \frac{P_4\,P_5}{P_1\,P_2\,P_3},$$

where

$$P_1 = -12(a^2+a+1)^2\wp(s) + (a+1)^2(a^4+1),$$
$$P_2 = 12(a^2+a+1)^2\wp(s) + (a+1)^2(2a^4-1),$$
$$P_3 = -12(a^2+a+1)^2\wp(s) + (a+1)^2(a^4-2),$$
$$P_4 = -144(a^2+a+1)^4\wp(s)^2 - 24(a+1)^2(a^2+a+1)^2(2a^4-1)\wp(s) + (a+1)^4(5a^8-5a^4-1),$$
$$P_5 = 144(a^2+a+1)^4\wp(s)^2 - 24(a+1)^2(a^2+a+1)^2(a^4-2)\wp(s) + (a+1)^4(a^8+5a^4-5).$$

The respective values $\wp(s)$ and $d(R_2/R_1)$ at the points corresponding to $\tilde{Q}_1$, $\tilde{Q}_2$ are given by

$$\wp = \frac{(a+1)^2(a^4+3a^2+1)}{12(a^2+a+1)^2} \quad \text{and} \quad d(R_2/R_1) = \frac{a^2+a}{a^2+a+1}.$$

The respective values $\wp(s)$ and $d(R_2/R_1)$ at the points corresponding to $\tilde{Q}_3$ and $\tilde{Q}_4$ are given by

$$\wp = \frac{(a+1)^2(a^4-3a^2+1)}{12(a^2+a+1)^2} \quad \text{and} \quad d(R_2/R_1) = -\frac{a^2+a}{a^2+a+1}.$$

For $a > 1$, we have

$$e_1 = \frac{(a+1)^2(a^4+1)}{12(a^2+a+1)^2} > e_2 = e_{1,3} = \frac{(a+1)^2(a^4-2)}{12(a^2+a+1)^2} > e_3 = e_{1,4} = -\frac{(a+1)^2(2a^4-1)}{12(a^2+a+1)^2},$$

and

$$Q_1 = \frac{3}{2}\omega_1, \quad Q_2 = \frac{1}{2}\omega_1, \quad Q_3 = \omega_2 + \frac{3}{2}\omega_1, \quad Q_4 = \omega_2 + \frac{1}{2}\omega_1.$$

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It follows that $Q_1 - Q_2 = Q_3 - Q_4 = \omega_1$, and hence,

$$\wp(Q_1 - Q_2) - e_1 = \wp(Q_3 - Q_4) - e_1 = 0.$$

Therefore, the off-diagonal entries $c_{12} = c_{21}$ and $c_{34} = c_{43}$ for the representation $\delta = 2$ become

$$c_{12} = c_{34} = 0.$$

We compute that

$$Q_1 - Q_4 = Q_1 + Q_3 = \omega_1 + \omega_2$$

corresponding to $\wp(\omega_1 + \omega_2) = e_2$. We also have

$$Q_1 - Q_3 = Q_1 + Q_4 = \omega_2$$

corresponding to $\wp(\omega_2) = e_3$. Similarly,

$$Q_2 - Q_3 = -(Q_1 - Q_4) = \omega_1 + \omega_2$$

and

$$Q_2 - Q_4 = -(Q_1 - Q_3) = \omega_2.$$

It follows that

$$\wp(Q_1 - Q_4) - e_1 = \wp(Q_2 - Q_3) - e_1 = e_2 - e_1 < 0$$

and

$$\wp(Q_1 - Q_3) - e_1 = \wp(Q_2 - Q_4) - e_1 = e_3 - e_1 < 0.$$

The off diagonal entries $c_{14}, c_{23}, c_{13}, c_{24}$ for $\delta = 2$ are computed by

$$c_{jk} = \epsilon(\beta_k - \beta_j)\frac{\sqrt{\wp(Q_k - Q_j) - e_1}}{\sqrt{-[(a^2+a)/(a^2+a+1)]^2}} = -i\,\epsilon(\beta - \beta_j)\sqrt{\wp(Q_k - Q_j) - e_1}\,\frac{a^2+a+1}{a^2+a},$$

and $c_{23} = \pm c_{14}$, $c_{24} = \pm c_{13}$. We obtain that

$$c_{13} = -i\,\epsilon((a-1)-(a+1))i\,\sqrt{e_1 - e_3}\,\frac{a^2+a+1}{a^2+a} = -2\epsilon\frac{a^2(a+1)}{2(a^2+a+1)}\,\frac{a^2+a+1}{a\,(a+1)} = -\epsilon\, a,$$

and

$$c_{14} = -i\,\epsilon(-(a-1)-(a+1))i\,\sqrt{e_1 - e_2}\,\frac{a^2+a+1}{a^2+a} = -i\,\epsilon(-2a)i\,\frac{(a+1)}{2(a^2+a+1)}\,\frac{a^2+a+1}{a(a+1)} = -\epsilon.$$

In conclusion, for $\delta = 2$ and $b = 1$, Theorem 2.1 produces the symmetric matrix

$$C_2 = \begin{pmatrix} 0 & 0 & a & 1 \\ 0 & 0 & 1 & a \\ a & 1 & 0 & 0 \\ 1 & a & 0 & 0 \end{pmatrix}.$$

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4. Conclusion. The Lax conjecture states that every hyperbolic ternary form admits a determinantal representation by real symmetric matrices. Recently, Helton and Vinnikov confirmed that the Lax conjecture is true by providing an explicit expression of the linear pencil of real symmetric matrices using Riemann theta functions with characteristics. It turns out the numerical range of a matrix is determined by the duality of the algebraic curve of the hyperbolic ternary form associated to the matrix. The determinantal representation provides various algebraic geometrical aspects and new methods to study the numerical ranges of matrices.

The formula of Helton-Vinnikov theorem requires evaluations of the Jacobi-variety and Riemann theta functions. We present a new method of constructing determinantal representations of elliptic curves via Weierstrass $\wp$-functions instead of Riemann theta functions. An example of this approach is provided. The main result gives an motivation to study new construction methods of the Helton-Vinnikov theorem which can be an interesting topic for further investigations.

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