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ON N/P–ASYMPTOTIC DISTRIBUTION OF VECTOR OF WEIGHTED TRACES OF POWERS OF WISHART MATRICES*

JOLANTA PIELASZKIEWICZ†, DIETRICH VON ROSEN‡, AND MARTIN SINGULL§

Abstract. The joint distribution of standardized traces of $\frac{1}{n}XX'$ and of $\left(\frac{1}{n}XX'\right)^2$, where the matrix $X : p \times n$ follows a matrix normal distribution is proved asymptotically to be multivariate normal under condition $\frac{n}{p} \overset{n,p\to\infty}{\to} c > 0$. Proof relies on calculations of asymptotic moments and cumulants obtained using a recursive formula derived in Pielaszkiewicz et al. (2015). The covariance matrix of the underlying vector is explicitely given as a function of n and p.

Key words. Wishart matrix, Multivariate normal distribution, Spectral distribution, Spectral moments, Covariance matrix.

AMS subject classifications. 62E20, 15B52, 62J10, 62M15.

1. Introduction. We continuously observe that the amount of data in a number of different applications becomes extensive. To make the analysis suitable for big data sets we make an assumption about the increasing sample size, i.e., $n \to \infty$, and the increasing dimensionality (number of parameters), i.e., $p \to \infty$. There is a strong interest to derive methods dealing with large–dimensional problems, such as problems related to large–size covariance matrices, when $n \geq p$, or $p > n$. The first of the mentioned cases has been widely studied as it corresponds to a natural case in which the sample size is bigger than the number of parameters. We can refer here to classical books on multivariate analysis such as [12] and [1]. The second of the mentioned cases has applications in, for example, finance, genetics and astronomy that are well–known examples of areas generating data sets with $p > n$. That case has been studied within the area of high–dimensional analysis, where, in particular, covariance estimation and regression have been investigated under $p > n$. Moreover, notice that in the case where $p > n$ the sample covariance matrix becomes singular, and hence cannot be inverted as demanded in a number of results valid for $p \leq n$. Problems under the assumption $p > n$ were, among others, considered by [13] (studies on the spectral distribution of specific classes of random matrices, when $p \to \infty$), [11] (a work of reference in random matrix theory) and [4] (studies on the distribution of covariance matrices and their eigenvalues, when $p \to \infty$). In this paper we assume that the Kolmogorov condition holds, which assures proportionality between p and n, as p and n tend to infinity, and is characterized by the constant c, such that $\lim_{n,p\to\infty} \frac{n}{p} = c \in (0, \infty)$. The assumption $c \in (0, \infty)$ allows us to obtain formulas depending on c that remain relevant for $p > n$ and $p \leq n$, although the elements of covariance matrix of the analyzed vector will be given directly as functions of sample size and number of parameters.

Let the trace $\mathrm{Tr}\{\cdot\}$ be defined as the sum of the diagonal elements of a square matrix and let $A^k = \underbrace{A \cdots A}_{k \text{ times}}$. Moreover, $\mathbb{E}[\cdot]$ denotes expectation. The trace of the power of a symmetric matrix relates to the sum of the

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powers of its eigenvalues. It is considered in multivariate analysis, for example as a tool while approximating densities, as well as in a number of hypothesis testing problems related to the covariance matrix. In this paper we also show applications of trace results to hypothesis testing. The normalized trace of powers of the Wishart matrix $W \sim \mathcal{W}_p(I, n)$, namely $\frac{1}{p}\operatorname{Tr}\{(\frac{1}{n}W)^k\}$, is an interesting object from several perspectives. It converges to its expectation, for any k, which can be derived using asymptotic results of [14]. Furthermore, the expectation of $\frac{1}{p}\operatorname{Tr}\{(\frac{1}{n}W)^k\}$ gives the kth moments of the Marchenko–Pastur distribution, see the original paper by [9] in high–dimensional analysis. In this paper we consider a scaled version of the normalized trace of powers of Wishart matrices, which is of the form

$$Y_k = \sqrt{np}\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^k\right\} - \mathbb{E}\left[\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^k\right\}\right]\right).$$

The expression will be proven to be central normally distributed for $k = 1$ and $k = 2$. Proofs can be extended to higher dimensions, i.e., for $k > 2$.

The distribution of $(\eta_1, \ldots, \eta_m)$, where

$$\eta_k = \frac{1}{(n+p)^k}\left(\operatorname{Tr}\{W^k\} - \mathbb{E}\left[\operatorname{Tr}\{W^k\}\right]\right)$$

has been considered by Jonsson, [6]. The random variables η_k and Y_k differ by a constant $\sqrt{c}/(1+c)$ in n/p–asymptotics. He provides a combinatorial proof based on moments calculations as suggested in [2] by Arharov. In the latter paper proofs are not given. Jonsson also improved scaling of the random variable, in relation to Arharov's paper, where $(n+p)^{k/2}\eta_k$ were considered. The proofs given in our paper are more straightforward than those given by Jonsson. See statement and proof of Theorem 4.1 regarding multivariate normal distribution in [6]. Moreover, in our paper the variance is given explicitly. In comparison, papers applying Jonsson's results, e.g., see [7], must use the results for moments given in [5], for example. In the present paper we give an alternative proof, using cumulants and moments of the variables Y_k seen as polynomials with respect to $\frac{1}{p}$. In this approach the coefficients of polynomials are determined in finite and asymptotic regime where the explicit calculations are made. The results are under symmetric scaling with respect to p and n [7] and give a covariance matrix that makes results suitable to be used in further applications, for example, in hypothesis testing.

The paper is organized as follows. In Section 2, notations are introduced, together with the recursive formula for $\mathbb{E}\big[\prod_{i=0}^k \operatorname{Tr}\{W^{m_i}\}\big]$, which will be used in a number of proofs of the paper. Then, in Section 3, we prove marginal asymptotic normality of Y_t, $t = 1, 2$. In Section 4 the joint asymptotic normality is considered and an explicit form of the covariance matrix is presented. In addition to conclusions presented in Section 5 we give some brief comments on application of results to hypothesis testing problem considered in paper [15].

2. Preliminaries. Let the matrix $X \in \mathbb{R}^{p\times n}$ follow a central matrix normal distribution, i.e., $X \sim \mathcal{N}_{p,n}(0, \Sigma, I_n)$. The matrix Σ denotes the dispersion matrix assumed to be positive definite and I_p stands for the identity matrix of size p. Then $W = XX'$, where X' denotes the transpose of X, follows a Wishart distribution, $W \sim \mathcal{W}_p(\Sigma, n)$ with n degrees of freedom. Moreover, as mentioned in the introduction, for all results in the paper we assume that the Kolmogorov condition holds, i.e., $\lim_{n,p\to\infty}\frac{n}{p} = c > 0$.

Let $W \sim \mathcal{W}_p(I_p, n)$. Then, by a result of [14], for all $k \in \mathbb{N}$ and all $m_0, m_1, \ldots, m_k$ such that $m_0 = 0$, $m_k \in \mathbb{N}$, $m_i \in \mathbb{N}_0$, $i = 1, \ldots, k-1$, we have the recursive relation for the expectation of the product of traces

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of powers of Wishart matrices

$$\begin{aligned}\mathbb{E}\Big[\prod_{i=0}^{k}\mathrm{Tr}\{W^{m_i}\}\Big] &= (n-p+m_k-1)\mathbb{E}\Big[\mathrm{Tr}\{W^{m_k-1}\}\prod_{i=0}^{k-1}\mathrm{Tr}\{W^{m_i}\}\Big]\\ &\quad+2\sum_{i=1}^{k-1}m_i\mathbb{E}\Big[\mathrm{Tr}\{W^{m_k+m_i-1}\}\prod_{\substack{j=0\\ j\neq i}}^{k-1}\mathrm{Tr}\{W^{m_j}\}\Big]\\ &\quad+\sum_{i=0}^{m_k-1}\mathbb{E}\Big[\mathrm{Tr}\{W^{i}\}\,\mathrm{Tr}\{W^{m_k-1-i}\}\prod_{j=0}^{k-1}\mathrm{Tr}\{W^{m_j}\}\Big]. \end{aligned}\tag{2.1}$$

We use $m_1^{(t)}(n,p)$ to denote $\mathbb{E}[\frac{1}{p}\,\mathrm{Tr}\{(\frac{1}{n}W)^t\}]$, in particular

$$\begin{aligned} m_1^{(1)}(n,p) &= \mathbb{E}\Big[\tfrac{1}{p}\,\mathrm{Tr}\Big\{\tfrac{1}{n}W\Big\}\Big]=1,\\ m_1^{(2)}(n,p) &= \mathbb{E}\Big[\tfrac{1}{p}\,\mathrm{Tr}\Big\{\Big(\tfrac{1}{n}W\Big)^2\Big\}\Big]=1+\tfrac{p}{n}+\tfrac{1}{n},\\ m_1^{(3)}(n,p) &= \mathbb{E}\Big[\tfrac{1}{p}\,\mathrm{Tr}\Big\{\Big(\tfrac{1}{n}W\Big)^3\Big\}\Big]=\Big(1+\tfrac{p}{n}+\tfrac{2}{n}\Big)\Big(1+\tfrac{p}{n}+\tfrac{1}{n}\Big)+\tfrac{p}{n}+\tfrac{2}{n^2},\\ m_1^{(4)}(n,p) &= \mathbb{E}\Big[\tfrac{1}{p}\,\mathrm{Tr}\Big\{\Big(\tfrac{1}{n}W\Big)^4\Big\}\Big]=\Big(1+\tfrac{p}{n}+\tfrac{3}{n}\Big)\Big((1+\tfrac{p}{n}+\tfrac{2}{n})(1+\tfrac{p}{n}+\tfrac{1}{n})+\tfrac{p}{n}+\tfrac{2}{n^2}\Big)\\ &\quad+2\Big(1+\tfrac{p}{n}+\tfrac{1}{n}\Big)\Big(\tfrac{p}{n}+\tfrac{4}{n^2}\Big). \end{aligned}$$

In the following section a vector $Y=(Y_1,Y_2)$ will be considered, where the tth coordinate is given by $Y_t=\sqrt{np}\Big(\frac{1}{p}\,\mathrm{Tr}\Big\{\Big(\frac{1}{n}W\Big)^t\Big\}-m_1^{(t)}(n,p)\Big)$, $t=1,2$ and $W\sim\mathcal{W}_p(I_p,n)$ and we find the asymptotic marginal distribution of Y_1 and Y_2. Note that the vector Y is centralized and standardized with $\sqrt{np}$ to ensure that its variance will not vanish in the limit.

3. Main results on asymptotic marginal distribution. The aim of this section is to prove theorems regarding the marginal normal distribution of Y_1 and Y_2, using formula (2.1). In Theorem 3.1 we consider $Y_1=\sqrt{np}\Big(\frac{1}{p}\,\mathrm{Tr}\Big\{\frac{1}{n}W\Big\}-1\Big)$. Here we can avoid (2.1) thanks to the fact that the moments of Y_1 can be computed using the χ^2–distribution. Hence, its moments are given by

$$\mathbb{E}[(\mathrm{Tr}\{W\})^k]=np(np+2)(np+4)\cdot\ldots\cdot(np+2(k-1)),$$

which can be calculated directly from the density function or the moment generating function. It leads us to the same equation as in (3.3) in the proof of Theorem 3.1 given below. Despite this fact, the proof is carried out so the reader is introduced to the methodology that will cover Y_2 (see, Theorem 3.6). Indeed the next theorem can also be proved via an expansion of the characteristic function.

THEOREM 3.1. *Let $W\sim\mathcal{W}_p(I_p,n)$. Then, under the Kolmogorov condition $\frac{n}{p}\overset{p,n\to\infty}{\to}c$, the asymptotic distribution of the random variable*

$$Y_1=\sqrt{np}\Big(\frac{1}{p}\,\mathrm{Tr}\Big\{\frac{1}{n}W\Big\}-1\Big)$$

is Gaussian with mean and variance equal to 0 and 2, respectively.

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Proof. Obviously,

$$\mathbb{E}[Y_1] = \sqrt{np}\,\mathbb{E}\Big[\tfrac{1}{p}\operatorname{Tr}\Big\{\tfrac{1}{n}W\Big\} - 1\Big] = \sqrt{np}\Big(\mathbb{E}\Big[\tfrac{1}{p}\operatorname{Tr}\Big\{\tfrac{1}{n}W\Big\}\Big] - 1\Big) = 0.$$

The result regarding the variance of Y_1 will follow from the proof of normality of the distribution since the proof is based on identification of moments (cumulants). The aim is to show that there are only a finite number of non-zero cumulants of Y_1, which is a characterization of the Gaussian distribution and was originally proved by Marcinkiewicz in [10]. The result of [10] was later referred to in a number of publications, including a paper by [3] and some recent papers, such as the one by [8]. To determine cumulants, the moments of Y_1 will be studied at first.

Using the formula in (2.1) for the expectation of the product of traces of Wishart matrices we obtain that for $m \geq 2$

$$\begin{aligned}
\mathbb{E}[Y_1^m] &= \mathbb{E}\Big[(np)^{m/2}\Big(\tfrac{1}{p}\operatorname{Tr}\Big\{\tfrac{1}{n}W\Big\} - m_1^{(1)}(n,p)\Big)^m\Big] = (np)^{m/2}\mathbb{E}\Big[\Big(\tfrac{1}{p}\operatorname{Tr}\Big\{\tfrac{1}{n}W\Big\} - 1\Big)^m\Big] \\
&= (np)^{m/2}\sum_{i=0}^{m}\tbinom{m}{i}(-1)^{m-i}\mathbb{E}\Big[\Big(\tfrac{1}{p}\operatorname{Tr}\Big\{\tfrac{1}{n}W\Big\}\Big)^i\Big] \\
&= (np)^{m/2}\Big[(-1)^m + \sum_{i=1}^{m}\tbinom{m}{i}(-1)^{m-i}\Big\{\Big(1 + \tfrac{2(i-1)}{np}\Big)\mathbb{E}\Big[\Big(\tfrac{1}{p}\operatorname{Tr}\Big\{\tfrac{1}{n}W\Big\}\Big)^{i-1}\Big]\Big\}\Big] \\
&= (np)^{m/2}\Big[(-1)^m + \sum_{i=1}^{m}\tbinom{m}{i}(-1)^{m-i}\prod_{l=0}^{i-1}\Big(1 + \tfrac{2l}{np}\Big)\Big].
\end{aligned} \tag{3.3}$$

We rewrite this formula as a polynomial with respect to $\frac{1}{np}$ and will determine the coefficients of the polynomial, i.e.,

$$\begin{aligned}
\mathbb{E}[Y_1^m] &= (np)^{m/2}\Big(a_{1,m} + a_{2,m}\tfrac{2}{np} + a_{3,m}\Big(\tfrac{2}{np}\Big)^2 + \ldots + a_{m,m}\Big(\tfrac{2}{np}\Big)^{m-1}\Big) \\
&= a_{1,m}(np)^{m/2} + a_{2,m}(np)^{m/2-1} + a_{3,m}2^2(np)^{m/2-2} + \ldots + a_{m,m}2^{m-1}(np)^{1-m/2},
\end{aligned}$$

where

$$\begin{aligned}
a_{1,m} &:= \sum_{i=0}^{m}(-1)^{m-i}\tbinom{m}{i}, \quad \text{for} \quad m \geq 1, \\
a_{2,m} &:= \sum_{i=2}^{m}(-1)^{m-i}\tbinom{m}{i}\sum_{k=1}^{i-1}k = \frac{m}{2}\sum_{i=1}^{m-1}(-1)^{m-i-1}i\tbinom{m-1}{i}, \quad \text{for} \quad m \geq 2, \\
a_{3,m} &:= \sum_{i=3}^{m}(-1)^{m-i}\tbinom{m}{i}\sum_{k=2}^{i-1}k\sum_{w=1}^{k-1}w, \quad \text{for} \quad m \geq 3, \\
&\vdots \\
a_{k,m} &:= \sum_{i=k}^{m}(-1)^{m-i}\tbinom{m}{i}\sum_{j_1=k-1}^{i-1}j_1\sum_{j_2=k-2}^{j_1-1}j_2\ldots\sum_{j_{k-1}=1}^{j_{k-2}-1}j_{k-1}, \quad \text{for} \quad m \geq k, \qquad (3.4) \\
&\vdots \\
a_{m,m} &:= \sum_{i=m}^{m}(-1)^{m-i}\tbinom{m}{i}\sum_{j_1=m-1}^{i-1}j_1\sum_{j_2=m-2}^{j_1-1}j_2\ldots\sum_{j_{m-1}=1}^{j_{m-2}-1}j_{m-1} = (m-1)!.
\end{aligned}$$

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Note that the coefficients $a_{k,m}$ are only defined for $k \leq m$ and that $a_{m,m} = (m-1)!$.

To be able to show that the odd moments vanish, and even moments have some particular form, we prove the following two lemmas.

Lemma 3.2. *A coefficient $a_{k,m}$, given in (3.4), satisfies the following recursive formula*

$$a_{k,m} = (m-1)! \sum_{i=2}^{k-1} \frac{1}{(m-i)!} a_{k-(i-1),m-i}$$

for any $k \geq 3$ and any $m > k$.

Proof. To prove the lemma we use the definition of $a_{k,m}$ and some standard calculations. We have

$$\begin{aligned}
a_{k,m} &= \sum_{i=k}^{m} (-1)^{m-i} \binom{m}{i} \sum_{j_1=k-1}^{i-1} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \\
&= (-1)^{m-k} \binom{m}{k} (k-1)! + \sum_{i=k+1}^{m} (-1)^{m-i} \binom{m}{i} \sum_{j_1=k-1}^{i-1} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \\
&= (-1)^{m-k} \binom{m}{k} (k-1)! + (-1)^{m-(k+1)} \binom{m}{k+1} \sum_{j_1=k-1}^{k} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \\
&\quad + (-1)^{m-(k+2)} \binom{m}{k+2} \sum_{j_1=k-1}^{k+1} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} + \ldots \\
&\quad + (-1)^{m-m} \binom{m}{m} \sum_{j_1=k-1}^{m-1} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \\
&= \left((-1)^{m-k} \binom{m}{k} + (-1)^{m-(k+1)} \binom{m}{k+1} + (-1)^{m-(k+2)} \binom{m}{k+2} + \ldots + 1 \right) (k-1)! \\
&\quad + (-1)^{m-(k+1)} \binom{m}{k+1} \sum_{j_1=k}^{k} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \\
&\quad + (-1)^{m-(k+2)} \binom{m}{k+2} \sum_{j_1=k}^{k+1} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \\
&\quad + \ldots + \sum_{j_1=k}^{m-1} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \\
&= (-1)^{m-k} \binom{m-1}{m-k} (k-1)! + (-1)^{m-(k+1)} \binom{m}{k+1} k \sum_{j_2=k-2}^{k-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \\
&\quad + (-1)^{m-(k+2)} \binom{m}{k+2} \left[k \sum_{j_2=k-2}^{k-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} + (k+1) \sum_{j_2=k-2}^{k} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \right] \\
&\quad + \ldots + \left[k \sum_{j_2=k-2}^{k-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} + \sum_{j_1=k+1}^{m-1} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \right]
\end{aligned}$$

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$$= (-1)^{m-k}\binom{m-1}{m-k}(k-1)! + (-1)^{m-(k+1)}\binom{m-1}{m-(k+1)} k \sum_{j_2=k-2}^{k-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}$$
$$+ (-1)^{m-(k+2)}\binom{m}{k+2}(k+1) \sum_{j_2=k-2}^{k} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} + \ldots + \sum_{j_1=k+1}^{m-1} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}$$
$$= (-1)^{m-k}\binom{m-1}{m-k}(k-1)! + (-1)^{m-(k+1)}\binom{m-1}{m-(k+1)} k \sum_{j_2=k-2}^{k-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}$$
$$+ (-1)^{m-(k+2)}\binom{m-1}{m-(k+2)}(k+1) \sum_{j_2=k-2}^{k} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}$$
$$+ (-1)^{m-(k+3)}\binom{m}{k+3}(k+2) \sum_{j_2=k-2}^{k+1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} + \ldots + \sum_{j_1=k+3}^{m-1} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}$$
$$= (-1)^{m-k}\binom{m-1}{m-k}(k-1)! + (-1)^{m-(k+1)}\binom{m-1}{m-(k+1)} k \sum_{j_2=k-2}^{k-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}$$
$$+ (-1)^{m-(k+2)}\binom{m-1}{m-(k+2)}(k+1) \sum_{j_2=k-2}^{k} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}$$
$$+ \ldots + (-1)^{0}\binom{m-1}{0}(m-1) \sum_{j_2=k-2}^{m-2} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}$$
$$= (-1)^{m-k}\binom{m-1}{m-k}(k-1)! + (m-1)\Bigg[(-1)^{m-(k+1)}\binom{m-2}{m-(k+1)} \sum_{j_2=k-2}^{k-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}$$
$$+ (-1)^{m-(k+2)}\binom{m-2}{m-(k+2)} \sum_{j_2=k-2}^{k} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}$$
$$+ \ldots + (-1)^{0}\binom{m-2}{0} \sum_{j_2=k-2}^{m-2} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}\Bigg]$$
$$= (-1)^{m-k}\binom{m-1}{m-k}(k-1)! + (m-1)\Bigg[\sum_{i=k-1}^{m-2} (-1)^{m-2-i}\binom{m-2}{i} \sum_{j_2=k-2}^{i} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}\Bigg]$$
$$= (-1)^{m-k}\binom{m-1}{m-k}(k-1)! + (m-1)\Bigg[\underbrace{\sum_{i=k-1}^{m-2} (-1)^{m-2-i}\binom{m-2}{i} \sum_{j_2=k-2}^{i-1} j_2 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}}_{a_{k-1,m-2}}$$
$$+ \sum_{i=k-1}^{m-2} (-1)^{m-2-i}\binom{m-2}{i} i \sum_{j_3=k-3}^{i-1} j_3 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}\Bigg]$$
$$= (-1)^{m-k}\binom{m-1}{m-k}(k-1)!$$
$$+ (m-1)\Bigg[a_{k-1,m-2} + (m-2) \sum_{i=k-1}^{m-2} (-1)^{m-2-i}\binom{m-3}{i-1} \sum_{j_3=k-3}^{i-1} j_3 \ldots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}\Bigg]$$

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$$
\begin{aligned}
&= (-1)^{m-k}\binom{m-1}{m-k}(k-1)! \\
&\quad + (m-1)\Bigg[a_{k-1,m-2} + (m-2)\sum_{i=k-2}^{m-3}(-1)^{m-3-i}\binom{m-3}{i}\sum_{j_3=k-3}^{i} j_3 \dots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}\Bigg] \\
&= (-1)^{m-k}\binom{m-1}{m-k}(k-1)! + \frac{(m-1)!}{(m-k)!}(-1)^{m-k+1} + \sum_{i=2}^{k-1}\frac{(m-1)!}{(m-i)!}a_{k-(i-1),m-i} \\
&= (m-1)!\sum_{i=2}^{k-1}\frac{1}{(m-i)!}a_{k-(i-1),m-i},
\end{aligned}
$$

which establishes the lemma. ∎

Moreover, in the following lemma we show that for all moments $\mathbb{E}[Y_1^m]$ there are only a finite number of coefficients $a_{k,m}$ that are different from zero.

LEMMA 3.3. *For the coefficient $a_{k,m}$, given in (3.4), it holds that*

$$a_{k,m} = 0 \qquad \textit{if} \qquad m \geq 2k-1$$

for all m and k, such that $m \geq k$.

Proof. Obviously, for all m

$$a_{1,m} = \sum_{i=0}^{m}(-1)^{m-i}\tbinom{m}{i} = (1-1)^m = 0.$$

We now show that $a_{2,m}$ is equal to zero for $m = 3, 4, 5, \dots$ so the only non-zero coefficient $a_{2,m}$ is $a_{2,2} = 1$. For $m > 2$

$$
\begin{aligned}
a_{2,m} &= \frac{m}{2}\sum_{i=1}^{m-1}(-1)^{m-i-1}i\binom{m-1}{i} \\
&\overset{m\geq 3}{=} \frac{m}{2}\left[m-1+\sum_{i=1}^{m-2}(-1)^{m-i-1}i\left(\binom{m-2}{i-1}+\binom{m-2}{i}\right)\right] \\
&= \frac{m}{2}\left[m-1+\sum_{i=1}^{m-2}(-1)^{m-i-1}i\binom{m-2}{i-1}+\sum_{i=1}^{m-2}(-1)^{m-i-1}i\binom{m-2}{i}\right] \\
&= \frac{m}{2}\left[\sum_{i=1}^{m-1}(-1)^{m-i-1}i\binom{m-2}{i-1}+\sum_{i=1}^{m-2}(-1)^{m-i-1}i\binom{m-2}{i}\right] \\
&= \frac{m}{2}\left[\sum_{i=0}^{m-2}(-1)^{m-i}(i+1)\binom{m-2}{i}+\sum_{i=1}^{m-2}(-1)^{m-i-1}i\binom{m-2}{i}\right] \\
&= \frac{m}{2}\left[\sum_{i=1}^{m-2}(-1)^{m-i}i\binom{m-2}{i}+\sum_{i=0}^{m-2}(-1)^{m-i}\binom{m-2}{i}-\sum_{i=1}^{m-2}(-1)^{m-i}i\binom{m-2}{i}\right] \\
&= \frac{m}{2}\sum_{i=0}^{m-2}(-1)^{m-i}\binom{m-2}{i} = 0.
\end{aligned}
$$

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Moreover, we show that $a_{k,k+1} = \sum_{i=2}^{k-1} \frac{k!}{i}$ for $k \geq 3$. We have

$$\begin{aligned}
a_{k,k+1} &= \sum_{i=k}^{k+1} (-1)^{k+1-i} \binom{k+1}{i} \sum_{j_1=k-1}^{i-1} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \dots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \\
&= -\binom{k+1}{k}(k-1)! + \sum_{j_1=k-1}^{k} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \dots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} \\
&= -(k+1)(k-1)! + \sum_{j_1=k-1}^{k} j_1 \sum_{j_2=k-2}^{j_1-1} j_2 \dots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} = -(k+1)(k-1)! + (k-1)! \\
&\quad + k \sum_{j_2=k-2}^{k-1} j_2 \dots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1} = -k! + k\left((k-2)! + (k-1) \sum_{j_3=k-3}^{k-2} j_3 \dots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}\right) \\
&= -k! + k(k-2)! + k(k-1)\left((k-3)! + (k-2) \sum_{j_4=k-4}^{k-3} j_4 \dots \sum_{j_{k-1}=1}^{j_{k-2}-1} j_{k-1}\right) \\
&= -k! + k(k-2)! + k(k-1)(k-3)! + \ldots + k(k-1)\cdots 3 \sum_{j_{k-1}=1}^{2} j_{k-1} = \sum_{i=2}^{k-1} \frac{k!}{i}.
\end{aligned}$$

Since $a_{k,m}$ is defined for $k \leq m$ and due to the fact that $a_{k,k} = (k-1)!$ and $a_{k,k+1} = \sum_{i=2}^{k-1} \frac{k!}{i}$ for $k \geq 3$, we always have at least two coefficients that do not vanish. Further we aim to prove that there are only $k-1$ non-zero coefficients in $\mathbb{E}[Y_1^k]$. Hence, $a_{k,m} = 0$ for $m \geq 2k-1$ remains to be shown. For this part of the proof, mathematical induction will be used. As we already have seen $a_{1,m} = 0$, for all $m \geq 1$, and $a_{2,m} = 0$, for all $m \geq 3$, so the first inductive step is completed. We assume that $a_{i,m} = 0$ for all $m \geq 2i-1$, for $i = 1, 2, \ldots, k-1$. Then, for $m \geq 2k-1$

$$a_{k,m} = (m-1)! \sum_{i=2}^{k-1} \frac{1}{(m-i)!} a_{k-(i-1),m-i} = 0,$$

where the first equality is by Lemma 3.2 and the last equality follows from the inductive assumption, as for all $i = 2, \ldots, k-1$ the first index of $a_{k-(i-1),m-i}$ is a number between 2 and $k-1$ and $m-i \geq 2(k-(i-1))-1 = 2k-2i+1$ implies $m-2k+1 \geq 2-i$ which is true for all the possible values of i. Finally, by mathematical induction $a_{k,m} = 0$ for all $m \geq 2k-1$ and all $k \geq 3$. ∎

Having investigated the properties of the coefficients $a_{k,m}$, we prove Theorem 3.4. It states that the asymptotic value of the mth moment of Y_1 is zero for odd m, and is a finite number, specified in the theorem, for even moments.

THEOREM 3.4. *For any m and $Y_1 = \sqrt{np}\left(\frac{1}{p}\operatorname{Tr}\left\{\frac{1}{n}W\right\} - 1\right)$ the moments of Y_1 converge according to*

$$m_k^{Y_1} := \mathbb{E}[Y_1^k] \to \begin{cases} 0, & \text{if } k \text{ is odd}, \\ 2^{k/2} a_{k/2+1,k}, & \text{if } k \text{ is even}, \end{cases} \qquad \text{as } \frac{n}{p} \to c, \quad p \to \infty, n \to \infty.$$

Proof. Consider $\mathbb{E}[Y_1^k]$. We assume first that k is odd, i.e., $k = 2l+1$ for some $l \in \mathbb{N}$. Then, $a_{i,k} = 0$ for all $i = 1, \ldots, l+1$ by Lemma 3.3. Hence the first non-zero term in $\mathbb{E}[Y_1^k]$ equals

$$\sqrt{(np)^{2l+1}} a_{l+2,k} \left(\frac{2}{np}\right)^{l+1} = \frac{a_{l+2,k}}{\sqrt{np}} \to 0 \qquad \text{as } \frac{n}{p} \to c, \quad p \to \infty, n \to \infty.$$

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For even moments, i.e., $k = 2l$ for some $l \in \mathbb{N}$, we have $a_{i,k} = 0$ for all $i = 1, \dots, l$. Hence, the first non-zero term in the $\mathbb{E}[Y_1^k]$ is $\sqrt{(np)^{2l}}a_{l+1,k}\left(\frac{2}{np}\right)^l = 2^l a_{l+1,k}$. Finally, we obtain as stated in the theorem

$$\mathbb{E}[Y_1^k] \to \begin{cases} 0, & \text{if } k \text{ is odd}, \\ 2^{k/2}a_{k/2+1,k}, & \text{if } k \text{ is even}, \end{cases} \qquad \text{as} \qquad \frac{n}{p} \to c, \quad p \to \infty, n \to \infty.$$

Now, after having presented the above result about moments we continue by studying the cumulants of Y_1. We denote the kth cumulant of the random variable Y_1 by $c_k^{Y_1}$. We can use the classical moment–cumulant relation formula

$$c_k^{Y_1} = m_k^{Y_1} - \sum_{p=1}^{k-1} \binom{k-1}{k-p} c_p^{Y_1} m_{k-p}^{Y_1}. \tag{3.5}$$

Then, by Theorem 3.4, the first four cumulants converge to

$$\begin{aligned}
&c_1^{Y_1} = 0, \\
&c_2^{Y_1} = m_2^{Y_1} - c_1^{Y_1} m_1^{Y_1} = m_2^{Y_1} = 2a_{2,2} = 2, \\
&c_3^{Y_1} = m_3^{Y_1} - c_1^{Y_1} m_2^{Y_1} - 2c_2^{Y_1} m_1^{Y_1} \overset{n,p\to\infty}{\to} 0, \\
&c_4^{Y_1} = m_4^{Y_1} - c_1^{Y_1} m_3^{Y_1} - 3c_2^{Y_1} m_2^{Y_1} - 3c_3^{Y_1} m_1^{Y_1} = m_4^{Y_1} - 3c_2^{Y_1} m_2^{Y_1} \overset{n,p\to\infty}{\to} 2^2 a_{3,4} - 3(2a_{2,2})^2 \\
&\quad = 4(a_{3,4} - 3a_{2,2}^2) = 0.
\end{aligned}$$

The variance $\mathbb{V}ar[Y_1] = c_2^{Y_1} = 2$ as stated in Theorem 3.1.

Obviously, all cumulants of odd order converge to zero, as both the odd moments are zero and the parity of moments and cumulants in the products of (3.5) are different (hence, one of them always goes to 0). Indeed, by the moment–cumulant formula we have

$$c_k^{Y_1} \to m_k^{Y_1} - \binom{k-1}{k-2} c_2^{Y_1} m_{k-2}^{Y_1} \overset{k \text{ is odd}}{=} 0.$$

Consider now the kth cumulant for even k, i.e., $k = 2l$. Let $k \geq 4$. Then, by Lemma 3.2 and Lemma 3.3

$$\begin{aligned}
&c_k^{Y_1} \to m_k^{Y_1} - \binom{k-1}{k-2} c_2^{Y_1} m_{k-2}^{Y_1} = m_{2l}^{Y_1} - (2l-1)2a_{2,2}2^{l-1}a_{l,2l-2} \\
&= 2^l a_{l+1,2l} - (2l-1)2^l a_{2,2} a_{l,2l-2} = 2^l(a_{l+1,2l} - (2l-1)a_{2,2}a_{l,2l-2}) \\
&= 2^l(a_{l+1,2l} - (2l-1)a_{l,2l-2}) = 2^l\Big((2l-1)! \sum_{i=2}^{l} \frac{1}{(2l-i)!} a_{l+2-i,2l-i} - (2l-1)a_{l,2l-2}\Big) \\
&= 2^l\Big((2l-1)! \frac{1}{(2l-2)!} a_{l,2l-2} - (2l-1)a_{l,2l-2}\Big) = 2^l(2l-1)\Big(a_{l,2l-2} - a_{l,2l-2}\Big) = 0.
\end{aligned}$$

Thus, we have proved that all cumulants of Y_1, apart from the second one, converge to zero, which is a characterization of the Gaussian distribution, see [10]. Hence, Y_1 converges to a Gaussian random variable which completes the proof of Theorem 3.1. □

Using the recursive formula (2.1), we will now prove the asymptotic distribution for Y_2. The difficulty in this case comes from the fact that the recursive formula we are using has a different form depending on

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the moment we are considering. Similarly to the proof of Theorem 3.1 we will show asymptotic normality by proving that there are only a finite number of cumulants that do not vanish. To be able to reach such a conclusion, we prove the following statement regarding the moments of Y_2.

THEOREM 3.5.

$$\mathbb{E}[Y_2^k] \rightarrow \begin{cases} 0, & \textit{if } k \textit{ is odd}, \\ d_k(c), & \textit{if } k \textit{ is even}, \end{cases} \qquad \textit{as} \qquad \frac{n}{p} \to c, \ \ p \to \infty, n \to \infty,$$

where $d_k(c)$ is a function of the constant c, which depends on k.

Proof. By a binomial expansion

$$\begin{aligned}
\mathbb{E}[Y_2^m] &= \mathbb{E}\Big[(np)^{m/2}\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\} - m_1^{(2)}(n,p)\Big)^m\Big] \\
&= (np)^{m/2}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\mathbb{E}\Big[\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^i\Big]\Big(\mathbb{E}\Big[\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big]\Big)^{m-i}.
\end{aligned}$$

We consider the sum

$$\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\mathbb{E}\Big[\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^i\Big]\Big(\mathbb{E}\Big[\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big]\Big)^{k-i} \tag{3.6}$$

as a polynomial with respect to $\frac{1}{p}$, after the substitution $n = ph$, where $h \to c$ for increasing n and p. Using the formula (2.1) for any k we have

$$\begin{aligned}
&\Big(\mathbb{E}\Big[\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{ph}W\Big)^2\Big\}\Big]\Big)^{k-i}\mathbb{E}\Big[\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{ph}W\Big)^2\Big\}\Big)^i\Big] \\
&\quad = \Big(1+\frac{1}{h}+\frac{1}{hp}\Big)^{k-i+1}\Big(1+\frac{4(i-1)}{hp^2}\Big)\mathbb{E}\Big[\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{i-1}\Big] \\
&\qquad + \frac{4(i-1)}{hp^2}\Big(1+\frac{1}{h}+\frac{1}{hp}\Big)^{k-i}\mathbb{E}\Big[\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^3\Big\}\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{k-2}\Big] \\
&\quad = \Big(1+\frac{1}{h}+\frac{1}{hp}\Big)^k + O\Big(\frac{1}{p^2}\Big) = \Big(1+\frac{1}{h}\Big)^k + \frac{k}{hp}\Big(1+\frac{1}{h}\Big)^{k-1} + O\Big(\frac{1}{p^2}\Big).
\end{aligned}$$

Hence, the constant term and the term of order $\frac{1}{p}$ of (3.6), given as a polynomial with respect to $\frac{1}{p}$, are equal to zero. Indeed,

$$\begin{aligned}
&\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\Big\{\Big(1+\frac{1}{h}\Big)^k + \frac{1}{hp}\Big(1+\frac{1}{h}\Big)^{k-1} + O\Big(\frac{1}{p^2}\Big)\Big\} \\
&\quad = \Big\{\Big(1+\frac{1}{h}\Big)^k + \frac{1}{hp}\Big(1+\frac{1}{h}\Big)^{k-1}\Big\}\underbrace{\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}}_{=0} + \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}O\Big(\frac{1}{p^2}\Big) \\
&\quad = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}O\Big(\frac{1}{p^2}\Big),
\end{aligned}$$

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where we mark the sum of alternating binomial coefficients that equals zero. Similarly, for the next term $\frac{1}{p^2}$ we identify the corresponding coefficient. The first two summands of (3.6), i.e., the cases when $i \in \{0,1\}$ can be written as

$$\left(\mathbb{E}\left[\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{ph}W\right)^2\right\}\right]\right)^{k-i}\mathbb{E}\left[\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{ph}W\right)^2\right\}\right)^i\right] \overset{i\in\{0,1\}}{=} \left(1+\frac{1}{h}+\frac{1}{ph}\right)^k$$

and hence the term of order $\frac{1}{p}$ equals

$$\binom{k}{2}\left(1+\frac{1}{h}\right)^{k-2}\frac{1}{h^2}, \qquad i = 0,1. \tag{3.7}$$

Depending on the index i, the coefficients of $\frac{1}{p^k}$, for $k \geq 2$, have a somewhat complicated form and will be further considered. In further investigation we assume that $i \geq 2$ and then

$$\begin{aligned}
&\left(\mathbb{E}\left[\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{hh}W\right)^2\right\}\right]\right)^{k-i}\mathbb{E}\left[\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{hh}W\right)^2\right\}\right)^i\right]\\
&\quad=\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i+1}\left(1+\frac{4(i-1)}{hp^2}\right)\mathbb{E}\left[\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^2\right\}\right)^{i-1}\right]\\
&\qquad+\frac{4(i-1)}{hp^2}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i}\mathbb{E}\left[\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^3\right\}\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^2\right\}\right)^{i-2}\right]\\
&\quad=\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i+1}\mathbb{E}\left[\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^2\right\}\right)^{i-1}\right]\\
&\qquad+\frac{4(i-1)}{hp^2}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i+1}\mathbb{E}\left[\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^2\right\}\right)^{i-1}\right]\\
&\qquad+\frac{4(i-1)}{hp^2}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i}\mathbb{E}\left[\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^3\right\}\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^2\right\}\right)^{i-2}\right]\\
&\quad=\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i+2}\left(1+\frac{4(i-2)}{hp^2}\right)\mathbb{E}\left[\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^2\right\}\right)^{i-2}\right]\\
&\qquad+\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i+1}\frac{4(i-2)}{hp^2}\mathbb{E}\left[\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^3\right\}\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^2\right\}\right)^{i-3}\right]\\
&\qquad+\frac{4(i-1)}{hp^2}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i+1}\mathbb{E}\left[\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^2\right\}\right)^{i-1}\right]\\
&\qquad+\frac{4(i-1)}{hp^2}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i}\mathbb{E}\left[\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^3\right\}\left(\frac{1}{p}\operatorname{Tr}\left\{\left(\frac{1}{n}W\right)^2\right\}\right)^{i-2}\right]
\end{aligned}$$

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$$
\begin{aligned}
&= \left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i+2}\mathbb{E}\Big[\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{i-2}\Big] \\
&\quad + \frac{4(i-2)}{hp^2}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i+2}\mathbb{E}\Big[\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{i-2}\Big] \\
&\quad + \frac{4(i-1)}{hp^2}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i+1}\mathbb{E}\Big[\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{i-1}\Big] \\
&\quad + \left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i+1}\frac{4(i-2)}{hp^2}\mathbb{E}\Big[\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^3\Big\}\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{i-3}\Big] \\
&\quad + \left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i}\frac{4(i-1)}{hp^2}\mathbb{E}\Big[\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^3\Big\}\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{i-2}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k} + \left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i}\sum_{j=1}^{i-1}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{j-1}\frac{4(i-j)}{hp^2} \\
&\qquad\qquad \mathbb{E}\Big[\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^3\Big\}\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{i-j-1}\Big] \\
&\quad + \left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i}\sum_{j=1}^{i-1}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{j}\frac{4(i-j)}{hp^2}\mathbb{E}\Big[\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{i-j}\Big] \\
&= \left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k} + \left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i}\sum_{j=1}^{i-1}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{j-1}\frac{4(i-j)}{hp^2}\Big\{\left(1+\frac{1}{h}+\frac{2}{hp}\right) \\
&\qquad\qquad \mathbb{E}\Big[\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{i-j}\Big]\Big\} \\
&\quad + \left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i}\sum_{j=1}^{i-1}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{j-1}\frac{4(i-j)}{hp^2} \\
&\qquad \Big\{\frac{1}{h}\mathbb{E}\Big[\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)\Big\}\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)\Big\}\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{i-j}\Big] + O(\frac{1}{p^2})\Big\} \\
&\quad + \left(1+\frac{1}{h}+\frac{1}{hp}\right)^{k-i}\sum_{j=1}^{i-1}\left(1+\frac{1}{h}+\frac{1}{hp}\right)^{j}\frac{4(i-j)}{hp^2}\mathbb{E}\Big[\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\}\Big)^{i-j}\Big].
\end{aligned}
$$

The coefficient corresponding to the term $\frac{1}{p^2}$ equals

$$
\begin{aligned}
&\binom{k}{2}\left(1+\frac{1}{h}\right)^{k-2}\frac{1}{h^2} + \Big\{\left(1+\frac{1}{h}\right)^{k}\sum_{j=1}^{i-1}\frac{8(i-j)}{h} + \left(1+\frac{1}{h}\right)^{k-2}\sum_{j=1}^{i-1}\frac{4(i-j)}{h^2}\Big\} \\
&\quad = \binom{k}{2}\left(1+\frac{1}{h}\right)^{k-2}\frac{1}{h^2} + \frac{4}{h}\left(1+\frac{1}{h}\right)^{k-2}\left(2\left(1+\frac{1}{h}\right)^2+\frac{1}{h}\right)\sum_{j=1}^{i-1}(i-j), \qquad i \geq 2.
\end{aligned}
$$

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From the above presentation and (3.7), the term of order $\frac{1}{p^2}$ of equation (3.6) vanishes for $k \geq 3$ as

$$\begin{aligned}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\Big\{\binom{k}{2}\Big(1+\frac{1}{h}\Big)^{k-2}\frac{1}{h^2}+\frac{4}{h}\Big(1+\frac{1}{h}\Big)^{k-2}\Big(2\Big(1+\frac{1}{h}\Big)^2+\frac{1}{h}\Big)1_{\{i\geq 2\}}\sum_{j=1}^{i-1}(i-j)\Big\}\\ =\frac{4}{h}\Big(1+\frac{1}{h}\Big)^{k-2}\Big(2\Big(1+\frac{1}{h}\Big)^2+\frac{1}{h}\Big)\sum_{i=2}^{k}\binom{k}{i}(-1)^{k-i}\sum_{j=1}^{i-1}(i-j)\\ =\frac{4}{h}\Big(1+\frac{1}{h}\Big)^{k-2}\Big(2\Big(1+\frac{1}{h}\Big)^2+\frac{1}{h}\Big)\sum_{i=2}^{k}\binom{k}{i}(-1)^{k-i}\sum_{j=1}^{i-1}(i-j)=0,\end{aligned} \tag{3.8}$$

where $1_{\{i\geq 2\}}$ stands for the indicator of a set $\{i \geq 2\}$. Moreover, the absence of the term $\frac{1}{p^2}$ indicates that the first following odd term, in this case $\frac{1}{p^3}$, also vanishes. It is a consequence of the fact that to obtain terms of order $\frac{1}{p^j}$ and $\frac{1}{p^{j+1}}$, where j is even, the recursive formula (2.1) has to be used the same number of times. Indeed, the coefficient for $\frac{1}{p^3}$ is given by

$$\binom{k}{3}\Big(1+\frac{1}{h}\Big)^{k-3}\frac{1}{h^3}+\Big\{\frac{2k+1}{h^2}\Big(1+\frac{1}{h}\Big)^{k-1}+\frac{k-1}{h^3}\Big(1+\frac{1}{h}\Big)^{k-2}\Big\}\sum_{j=1}^{i-1}4(i-j), i\geq 2.$$

Hence, by the same argument as in (3.8), the coefficient corresponding to the term $\frac{1}{p^3}$ equals zero for $k \geq 3$.

The same procedure can be repeated in formula (3.6) for terms of higher orders. As for the terms $\frac{1}{p^2}$ and $\frac{1}{p^3}$, zero was obtained due to the fact that

$$\sum_{i=2}^{k}\binom{k}{i}(-1)^{k-i}\sum_{j=1}^{i-1}(i-j)=0$$

for $k \geq 3$. As $\sum_{i=2}^{k}\binom{k}{i}(-1)^{k-i}\sum_{j=1}^{i-2}(i-j)\sum_{w=1}^{i-j-1}(i-j-w)=0$, for $k \geq 5$ also the terms involving $\frac{1}{p^4}$ and $\frac{1}{p^5}$ cancel out. In general, the term corresponding to $\frac{1}{p^{2m}}$ as well as $\frac{1}{p^{2m+1}}$ vanishes for $k \geq 2m+1$ as

$$\sum_{i=m+1}^{k}\binom{k}{i}(-1)^{k-i}\prod_{q=1}^{m}\sum_{j_{q+1}=1}^{j_q-m+q-1}(j_q-j_{q+1})=0, \text{ for } k\geq 2m+1 \text{ with } j_1=i.$$

If we assume for a moment that k is even, then the term corresponding to $\frac{1}{p^k}$ cannot vanish in (3.6) as it is not true that $k \geq k+1$. However, the term corresponding to $\frac{1}{p^{k-2}}$ vanishes as $k \geq k-2+1 = k-1$ holds. The following term $\frac{1}{p^{k-1}}$ also vanishes for the same reason. Assuming k to be odd, the term corresponding to $\frac{1}{p^{k-1}}$ vanishes as $k \geq k$ obviously is true, and then the same holds for the following odd term $\frac{1}{p^k}$. Finally, all terms of orders 1 up to $\frac{1}{p^{k-1}}$ in (3.6) vanish in the case of even k, and terms of order 1 up to $\frac{1}{p^k}$ in the case of k being odd. This implies the statement of Theorem 3.5 saying that odd moments of Y_2 asymptotically equal zero, while even ones tend to the constant depending on c. ∎

THEOREM 3.6. *Let $W \sim \mathcal{W}_p(I_p, n)$. Then, under the Kolmogorov condition $\frac{n}{p} \overset{p,n\to\infty}{\to} c$, the asymptotic distribution of $Y_2 = \sqrt{np}\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^2\Big\} - m_1^{(2)}(n,p)\Big)$, is Gaussian with mean zero and non-zero finite variance.*

Proof. The random variable Y_2 is centered as

$$\mathbb{E}[Y_2]=\sqrt{np}\,\mathbb{E}\Big[\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)\Big\}-m_1^{(2)}(n,p)\Big]=\sqrt{np}\Big(\mathbb{E}\Big[\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)\Big\}\Big]-m_1^{(2)}(n,p)\Big)=0.$$

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We still must prove, using the statement of Theorem 3.5, that all cumulants of degree greater than 2 asymptotically equal zero.

The moment–cumulant relation formula (3.5), together with the fact given by Theorem 3.5 that all odd moments of Y_2 are asymptotically equal to zero indicates that all odd cumulants of Y_2 vanish while $n, p \to \infty$. Furthermore, we need to investigate the cumulants of even degree.

For the arbitrary t the kth cumulant $c_k^{Y_t}$ for $k \geq 3$ is given, using the moment–cumulant relation formula, by

$$\begin{aligned}
c_k^{Y_t} &= m_k^{Y_t} - \sum_{p=1}^{k-1} \binom{k-1}{k-p} c_p^{Y_t} m_{k-p}^{Y_t} = m_k^{Y_t} - \binom{k-1}{k-2} c_2^{Y_t} m_{k-2}^{Y_t} = m_k^{Y_t} - (k-1) m_2^{Y_t} m_{k-2}^{Y_t} \\
&= (np)^{k/2} \left(\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \mathbb{E}\left[\left(\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right)^i \right] \left(\mathbb{E}\left[\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right] \right)^{k-i} \right) \\
&\quad -(k-1) np \left(\mathbb{E}\left[\left(\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right)^2 \right] - \left(\mathbb{E}\left[\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right] \right)^2 \right) \\
&(np)^{k/2-1} \left(\sum_{i=0}^{k-2} \binom{k-2}{i} (-1)^{k-2-i} \mathbb{E}\left[\left(\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right)^i \right] \left(\mathbb{E}\left[\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right] \right)^{k-2-i} \right) \\
&= (np)^{k/2} \Bigg\{ \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \mathbb{E}\left[\left(\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right)^i \right] \left(\mathbb{E}\left[\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right] \right)^{k-i} \\
&\quad -(k-1) \left(\mathbb{E}\left[\left(\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right)^2 \right] - \left(\mathbb{E}\left[\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right] \right)^2 \right) \\
&\quad \sum_{i=0}^{k-2} \binom{k-2}{i} (-1)^{k-2-i} \mathbb{E}\left[\left(\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right)^i \right] \left(\mathbb{E}\left[\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^t \right\} \right] \right)^{k-2-i} \Bigg\}.
\end{aligned} \tag{3.9}$$

As k is even and $t = 2$, by Theorem 3.5 it is sufficient to consider terms of order $\frac{1}{p^k}$ in (3.6). Note that both minuend and subtrahend of cumulants in (3.9) are polynomials of the same degree. We have

$$\begin{aligned}
&\underbrace{\left(\mathbb{E}\left[\left(\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^2 \right\} \right)^2 \right] - \left(\mathbb{E}\left[\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^2 \right\} \right] \right)^2 \right)}_{= \frac{4}{hp^2} \left\{ \left(1 + \frac{1}{h} + \frac{1}{hp}\right)^2 + \left(1 + \frac{1}{h} + \frac{1}{hp}\right)\left(1 + \frac{1}{h} + \frac{2}{hp}\right) + \frac{1}{h}\left(1 + \frac{2}{hp^2}\right) \right\}} \\
&\times \underbrace{\sum_{i=0}^{k-2} \binom{k-2}{i} (-1)^{k-2-i} \mathbb{E}\left[\left(\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^2 \right\} \right)^i \right] \left(\mathbb{E}\left[\frac{1}{p} \operatorname{Tr}\left\{ \left(\frac{1}{n} W \right)^2 \right\} \right] \right)^{k-2-i}}_{= O\left(\frac{1}{p^{k-2}} \right)}
\end{aligned} \tag{3.10}$$

which after multiplication gives a polynomial of $O(\frac{1}{p^k})$ (in first phrase we have terms of at least order $\frac{1}{p^2}$, and from the second term of order at least $\frac{1}{p^{k-2}}$). From previously performed calculations in the proof of Theorem 3.5 we know that the $\frac{1}{p^k}$ coefficient of the minuend of (3.9), for even k, is given by

$$\sum_{i=\frac{k}{2}+1}^{k} \binom{k}{i} (-1)^{k-i} \prod_{q=1}^{\frac{k}{2}} \sum_{j_{q+1}=1}^{j_q - \frac{k}{2} + q - 1} (j_q - j_{q+1}) \neq 0,$$

since it is not true that $k \geq k+1$. Moreover, the coefficient corresponding to $\frac{1}{p^k}$ in the subtrahend of (3.9) depends

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on three sums:

$$\sum_{i=\frac{k-2}{2}+1}^{k-2}\binom{k-2}{i}(-1)^{k-2-i}\prod_{q=1}^{\frac{k-2}{2}}\sum_{j_{q+1}=1}^{j_q-\frac{k-2}{2}+q-1}(j_q-j_{q+1})\neq 0,\ \text{since } k-2<k-1,$$

$$\sum_{i=\frac{k-4}{2}+1}^{k-2}\binom{k-2}{i}(-1)^{k-2-i}\prod_{q=1}^{\frac{k-4}{2}}\sum_{j_{q+1}=1}^{j_q-\frac{k-4}{2}+q-1}(j_q-j_{q+1})=0,\ \text{since } k-2\geq k-3,$$

$$\sum_{i=\frac{k-6}{2}+1}^{k-2}\binom{k-2}{i}(-1)^{k-2-i}\prod_{q=1}^{\frac{k-6}{2}}\sum_{j_{q+1}=1}^{j_q-\frac{k-6}{2}+q-1}(j_q-j_{q+1})=0,\ \text{since } k-2\geq k-5.$$

The above sums come from the three possible ways to obtain terms of order $\frac{1}{p^k}$ from (3.10). Finally, as

$$\sum_{i=\frac{k}{2}+1}^{k}\binom{k}{i}(-1)^{k-i}\prod_{q=1}^{\frac{k}{2}}\sum_{j_{q+1}=1}^{j_q-\frac{k}{2}+q-1}(j_q-j_{q+1})$$
$$-(k-1)\sum_{i=\frac{k-2}{2}+1}^{k-2}\binom{k-2}{i}(-1)^{k-2-i}\prod_{q=1}^{\frac{k-2}{2}}\sum_{j_{q+1}=1}^{j_q-\frac{k-2}{2}+q-1}(j_q-j_{q+1})=0,$$

we can conclude that the term $\frac{1}{p^k}$ vanishes in the cumulant c_k. Hence, $c_k=p^kO(\frac{1}{p^{k+1}})\to 0$ for any k which is a characterization of the Gaussian distribution and, therefore, finishes the proof of Theorem 3.6. ∎

4. Main results on asymptotic multivariate normal distribution. Although the marginal density for $Y_t=\sqrt{np}\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^t\Big\}-m_1^{(t)}(n,p)\Big)$ for $t\in\{1,2\}$ has already been proved in the previous section, we still need to verify the asymptotic multivariate normal distribution of the random vector $Y=(Y_1,Y_2)$.

THEOREM 4.1. *Let $W\sim\mathcal{W}_p(I_p,n)$ and $Y=(Y_1,Y_2)$, where*

$$Y_t=\sqrt{np}\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^t\Big\}-m_1^{(t)}(n,p)\Big)$$

for $t=1,2$. Then, if $\frac{n}{p}\overset{p,n\to\infty}{\to}c$, Y asymptotically follows a multivariate normal distribution with mean zero and covariance matrix

$$\Sigma_Y=(\mathbb{C}ov(Y_i,Y_j))_{i,j=1}^m=\begin{pmatrix}2 & 4(1+\frac{1}{c})\\ 4(1+\frac{1}{c}) & \frac{4}{c}+8(1+\frac{1}{c})^2\end{pmatrix}.$$

Proof. We will only show the expression for Σ_Y. The theorem has already been stated by [6]. In principle we could have copied the calculations from the previous section and apply them to $a_1Y_1+a_2Y_2$, for arbitrary a_1 and a_2, but the approach is lengthy.

The exact value of the variance $\mathbb{V}ar[Y_1]=2$ comes from the statement of Theorem 3.1. Remaining variances and covariance are calculated according to (2.1) and are given by

$$\mathbb{V}ar[Y_2]=\mathbb{V}ar\Big[\sqrt{np}\Big(\tfrac{1}{p}\operatorname{Tr}\Big\{\Big(\tfrac{1}{n}W\Big)^2\Big\}-m_1^{(2)}(n,p)\Big)\Big]=np\mathbb{V}ar\Big[\tfrac{1}{p}\operatorname{Tr}\Big\{\Big(\tfrac{1}{n}W\Big)^2\Big\}\Big]$$
$$=np\Big(\mathbb{E}\Big[\Big(\tfrac{1}{p}\operatorname{Tr}\Big\{\Big(\tfrac{1}{n}W\Big)^2\Big\}\Big)^2\Big]-\Big(\mathbb{E}\Big[\tfrac{1}{p}\operatorname{Tr}\Big\{\Big(\tfrac{1}{n}W\Big)^2\Big\}\Big]\Big)^2\Big)$$

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$$
\begin{aligned}
&= np\Big(\mathbb{E}\Big[\tfrac{1}{p^2n^4}\operatorname{Tr}\{(W)^2\}\operatorname{Tr}\{(W)^2\}\Big] - \Big(1+\tfrac{p}{n}+\tfrac{1}{n}\Big)^2\Big)\\
&= np\Big(\tfrac{1}{p^2n^4}\Big((n+p+1)\mathbb{E}\Big[\operatorname{Tr}\{W\}\operatorname{Tr}\{(W)^2\}\Big] + 4\mathbb{E}\Big[\operatorname{Tr}\{(W)^3\}\Big]\Big) - \Big(1+\tfrac{p}{n}+\tfrac{1}{n}\Big)^2\Big)\\
&= np\Big(\tfrac{1}{p^2n^4}\Big(((n+p+1)(np+4)+4(n+p+2))\mathbb{E}\Big[\operatorname{Tr}\{(W)^2\}\Big]\\
&\qquad\qquad + 4\mathbb{E}\Big[\operatorname{Tr}\{W\}\operatorname{Tr}\{W\}\Big]\Big) - \Big(1+\tfrac{p}{n}+\tfrac{1}{n}\Big)^2\Big)\\
&= np\Big(\tfrac{1}{p^2n^4}\Big(((n+p+1)(np+4)+4(n+p+2))(n+p+1)np + 4(np+2)np\Big)\\
&\qquad\qquad - \Big(1+\tfrac{p}{n}+\tfrac{1}{n}\Big)^2\Big)\\
&= np\Big(\Big(\Big(1+\tfrac{p}{n}+\tfrac{1}{n}\Big)\tfrac{4}{np}+\tfrac{4}{np}\Big(1+\tfrac{p}{n}+\tfrac{2}{n}\Big)\Big)\Big(1+\tfrac{p}{n}+\tfrac{1}{n}\Big)+\tfrac{4}{n^2}\Big(1+\tfrac{2}{np}\Big)\Big)\\
&= 4\Big(\Big(1+\tfrac{p}{n}+\tfrac{1}{n}\Big)+\Big(1+\tfrac{p}{n}+\tfrac{2}{n}\Big)\Big)\Big(1+\tfrac{p}{n}+\tfrac{1}{n}\Big)+\tfrac{4p}{n}\Big(1+\tfrac{2}{np}\Big)\\
&\qquad\qquad \to 8\Big(1+\tfrac{1}{c}\Big)^2+\tfrac{4}{c},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{C}ov[Y_1,Y_2] &= np\Big(\mathbb{E}\Big[\tfrac{1}{p}\operatorname{Tr}\Big\{\Big(\tfrac{1}{n}W\Big)^2\Big\}\tfrac{1}{p}\operatorname{Tr}\Big\{\tfrac{1}{n}W\Big\}\Big]\\
&\qquad\qquad - \mathbb{E}\Big[\tfrac{1}{p}\operatorname{Tr}\Big\{\tfrac{1}{n}W\Big\}\Big]\mathbb{E}\Big[\tfrac{1}{p}\operatorname{Tr}\Big\{\Big(\tfrac{1}{n}W\Big)^2\Big\}\Big]\Big)\\
&= np\Big(\tfrac{1}{p^2n^3}\mathbb{E}\Big[\operatorname{Tr}\{(W)^2\}\operatorname{Tr}\{W\}\Big] - \Big(1+\tfrac{1}{c}+\tfrac{1}{n}\Big)\Big)\\
&= np\Big(\tfrac{1}{p^2n^3}(np+4)\mathbb{E}\Big[\operatorname{Tr}\{(W)^2\}\Big] - \Big(1+\tfrac{1}{c}+\tfrac{1}{n}\Big)\Big)\\
&= np\Big(\tfrac{1}{p^2n^3}(np+4)(n+p+1)np\Big] - \Big(1+\tfrac{1}{c}+\tfrac{1}{n}\Big)\Big) = 4\Big(1+\tfrac{p}{n}+\tfrac{1}{n}\Big)\\
&\qquad\qquad \to 4\Big(1+\tfrac{1}{c}\Big).
\end{aligned}
$$

□

Without the proof we present an extended version of Theorem 4.1.

Proposition 4.2. *Let $W \sim \mathcal{W}_p(I_p, n)$ and $Y = (Y_1, Y_2, Y_3)$, where*

$$Y_t = \sqrt{np}\Big(\frac{1}{p}\operatorname{Tr}\Big\{\Big(\frac{1}{n}W\Big)^t\Big\} - m_1^{(t)}(n,p)\Big)$$

for $t = 1, 2, 3$. Then, the 3-dimensional vector Y is an asymptotically multivariate normal distribution with mean zero and covariance matrix

$$
\begin{aligned}
\Sigma_Y &= (\mathbb{C}ov(Y_i,Y_j))_{i,j=1}^m \qquad (4.11)\\
&= \begin{pmatrix} 2 & 4(1+\frac{1}{c}) & 6((1+\frac{1}{c})^2+\frac{1}{c})\\ 4(1+\frac{1}{c}) & \frac{4}{c}+8(1+\frac{1}{c})^2 & 12(1+\frac{1}{c})((1+\frac{1}{c})^2+\frac{2}{c})\\ 6((1+\frac{1}{c})^2+\frac{1}{c}) & 12(1+\frac{1}{c})((1+\frac{1}{c})^2+\frac{2}{c}) & 24((1+\frac{1}{c})^2+\frac{1}{c})^2+\frac{42}{c}(1+\frac{1}{c})^2 \end{pmatrix},
\end{aligned}
$$

when $\frac{n}{p} \overset{p,n\to\infty}{\to} c$.

5. Conclusions. We have proved asymptotic multivariate normality of the vector of the standardized traces of $\frac{1}{n}XX'$ and of $\left(\frac{1}{n}XX'\right)^2$ under the Kolmogorov condition. It is an algebraic proof based on the recursive formula for $\mathbb{E}\big[\prod_{i=0}^{k}\operatorname{Tr}\{(XX')^{m_i}\}\big]$ that is an alternative to the combinatorial result of Jonsson,

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[6]. In this paper we present an explicit form for the covariance matrix of the underlying vector as a function of n and p. Normality of the considered vector was utilized for example in authors article on testing for identity of the covariance matrix using a goodness-of-fit approach.

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